Differential Geometry
Inspiring Talks Series

Spring 2020
Dr. Jorge Basilio

Areas of Parametrized Surfaces & Wormholes

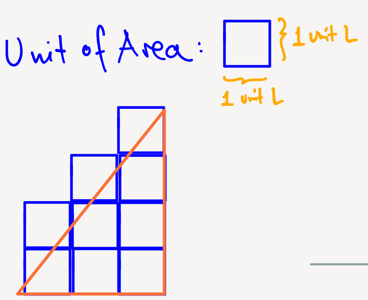
Objectives
- Review of Area
- Definition of Surface Area
- Computations and Examples
- Surfaces of Revolution
- Wormholes
- Addendum: Why is surface area a definition and not a theorem?

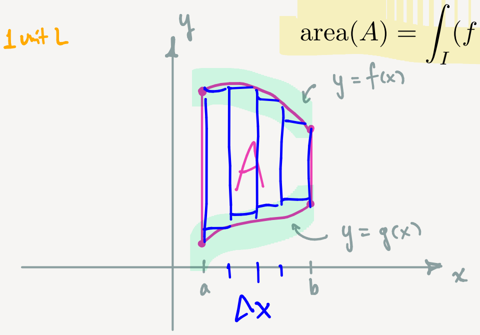
• Review of Area

• You might find it surprising that the concept of "area" has puzzled mathematicians since antiquity. Besides a rectangles and triangles and other highly symmetric shapes, it wasn't until the calculus was invented that the concept of area was generalized to more complicated regions.

• Planar Sets using Calculus

Given a subset $A \subset \mathbb{R}^2$ in the plane bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$, with $g(x) < f(x)$ and $a < b$, we can use the integral from Calc I:





$$\text{area}(A) = \int_I (f(x) - g(x))\,dx$$

• Riemann Sums:

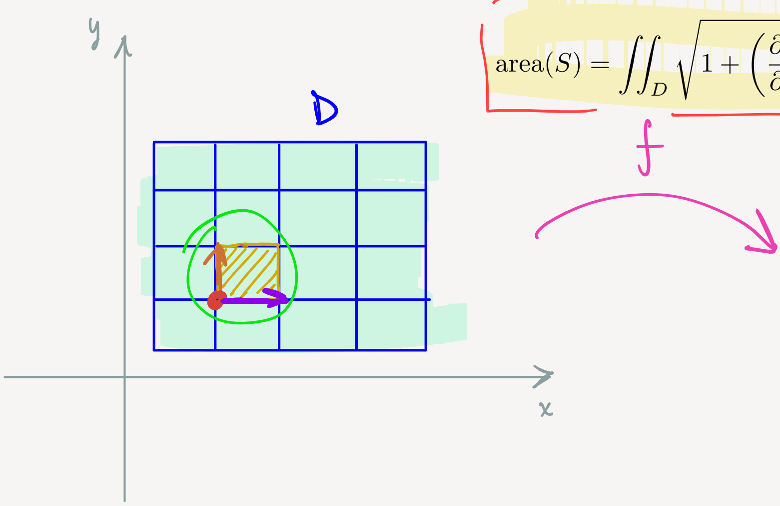
$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(f(c_i) - g(c_i)\right)\Delta x$$

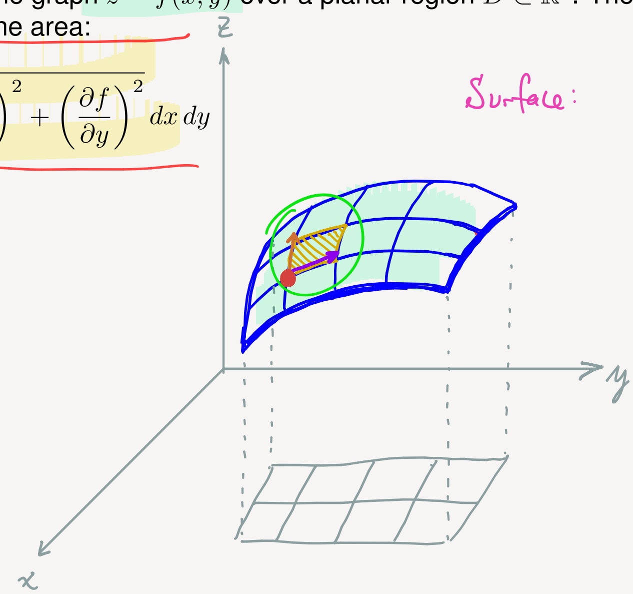
Remark The study of area in the plane is far from trivial! This problem was one of the major reasons calculus was invented. The story doesn't end there. Future intense study lead Lebesgue to generalize the theory of integration to include crazier sets that the Riemann integral couldn't handle. In fact, $\text{area}(A) = \iint_A dx\,dy$ suggests the connection runs deep and in analysis we can use measure theory to find the area of fractal like objects using $\text{area}(A) = \int_A d\mu$ where μ is the Lebesgue measure in $\mathbb{R}^2$.

• Graph of a two-variable function in Space using Calculus

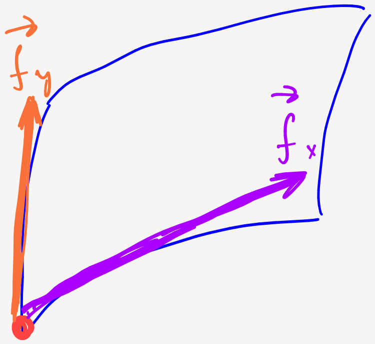
• Given a real-valued function of two variables, $f(x, y)$, we can study the graph $z = f(x, y)$ over a planar region $D \subset \mathbb{R}^2$. The set $S = \{(x, y, f(x, y)) \mid (x, y) \in D\}$ is a surface in $\mathbb{R}^3$ and we can define the area:

$$\boxed{\text{area}(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy}$$





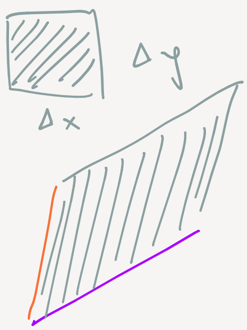
- Let's review why this is a reasonable definition for the surface area of S.



- $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = z$
- $\vec{f}_x = \langle 1, 0, f_x \rangle$ & $\vec{f}_y = \langle 0, 1, f_y \rangle$
- Area of parallelogram spanned by $\vec{f}_x$ & $\vec{f}_y$:

↳ $\|\vec{f}_x \times \vec{f}_y\| = \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} \right\| = \|\langle -f_x, -f_y, 1 \rangle\|$

$$= \sqrt{1 + (f_x)^2 + (f_y)^2}$$



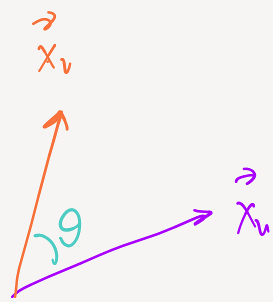
- Area of surface $\approx \|\vec{f}_x \times \vec{f}_y\| \Delta x \Delta y$

- **Re-interpretation**

When we view S using a patch: $\vec{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$,

$$\vec{x}(u,v) = \begin{pmatrix} x_1(u,v) \\ x_2(u,v) \\ x_3(u,v) \end{pmatrix},$$

or we can write this as a function $F : \mathbb{R}^2 \to \mathbb{R}^3$ with $F(u,v) = \langle f_1(u,v), f_2(u,v), f_3(u,v) \rangle$.



- Tangent vectors:

$\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$

$\vec{u} \cdot \vec{v} = \|u\| \|v\| \cos\theta$

$$\vec{x}_u = \begin{pmatrix} \frac{\partial x_1}{\partial u} \\ \frac{\partial x_2}{\partial u} \\ \frac{\partial x_3}{\partial u} \end{pmatrix} \quad \text{and} \quad \vec{x}_v = \begin{pmatrix} \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial v} \\ \frac{\partial x_3}{\partial v} \end{pmatrix}$$

- As we saw above, a little piece of area is approximated by $\|\vec{x}_u \times \vec{x}_v\| \, du \, dv$.
- Let's study $\|\vec{x}_u \times \vec{x}_v\|$ in a **new way**:

$$\|\vec{x}_u \times \vec{x}_v\|^2 = \left(\|\vec{x}_u\| \cdot \|\vec{x}_v\| \sin\theta\right)^2$$
$$= \|x_u\|^2 \cdot \|x_v\|^2 (1 - \cos^2\theta)$$
$$= (x_u \cdot x_u)(x_v \cdot x_v) - (x_u \cdot x_v)^2$$

- So what? Why did we do this?

$$= \det \begin{bmatrix} x_u \cdot x_u & x_u \cdot x_v \\ x_v \cdot x_u & x_v \cdot x_v \end{bmatrix} \leftarrow [g_{ij}] \text{ matrix}$$

- So, we just proved an interesting connection:

$$\boxed{\|\vec{x}_u \times \vec{x}_v\| = \sqrt{\det[g_{ij}]}}$$

Defn 1 First Fundamental Form

Let $M \subset \mathbb{R}^3$ be a smooth surface. Let $\vec{x} : U \subset \mathbb{R}^2 \to M$ be a regular patch. We define the **first fundamental form** of M, denoted by (g_{ij}), to be the tensor

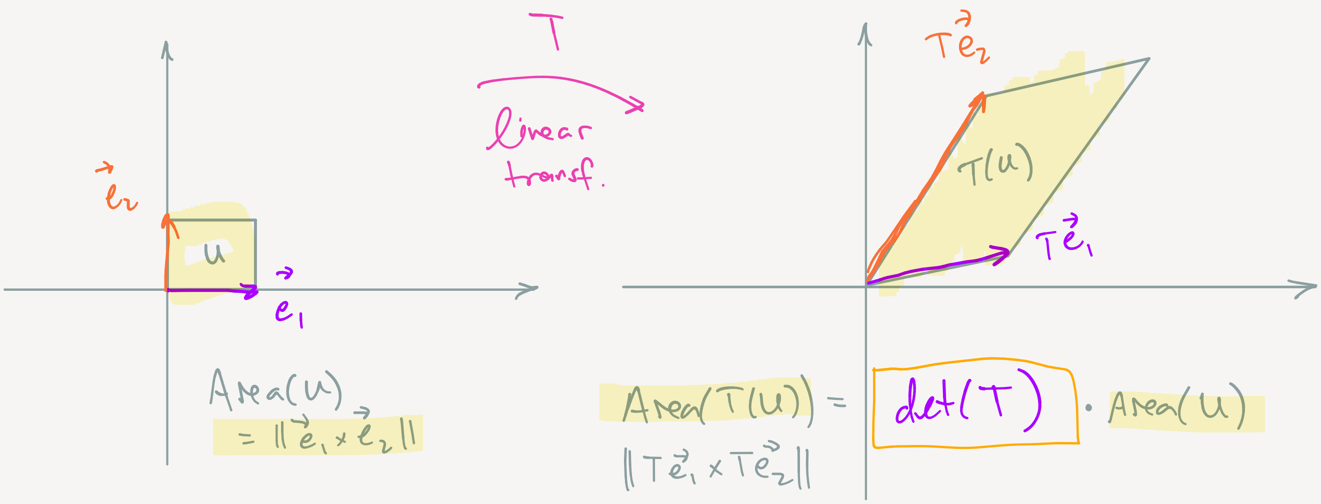
$$[\,g_{ij}\,] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \qquad \text{where} \qquad g_{ij} = \vec{x}_i \bullet \vec{x}_j \qquad \begin{bmatrix} \vec{x}_u \cdot \vec{x}_u & \vec{x}_u \cdot \vec{x}_v \\ \vec{x}_v \cdot \vec{x}_u & \vec{x}_v \cdot \vec{x}_v \end{bmatrix} \tag{1}$$

and the dot product is on $\mathbb{R}^3$.

Remarks So if $\vec{x}$ is regular (i.e. $\vec{x}_u$ and $\vec{x}_v$ are linearly independent at each point $(u, v) \in U$, then one can argue that $\det(g_{ij})$ is not zero. And conversely!
This is another way to check regularity instead of checking that $D\vec{x}$ has rank 2.

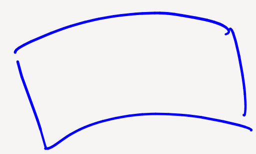
Why does the determinant show up?

- In linear algebra, the determinant is exactly the change in area of a parallelogram under the effect of a linear transformation.



- This will help us understand how to generalize the area formula to volumes and hyper-volumes in higher dimensions.

• Definition of Surface Area



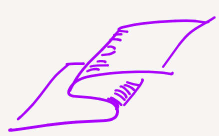
$M = \vec{x}(U)$

- We are now ready to define the surface area of a smooth manifold $M \subset \mathbb{R}^3$.

• Step 1

We start with the case where M can be described using a single patch: $\vec{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, and $M = \vec{x}(U)$.

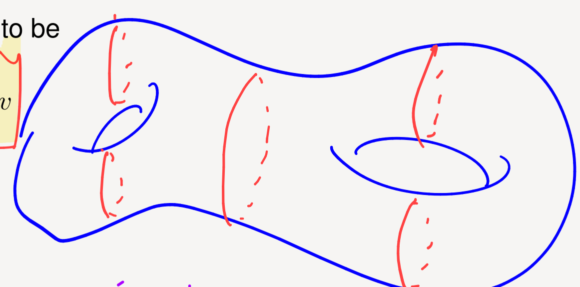
We've now seen that the local approximation to surface area, ΔS, is $\|\vec{x}_u \times \vec{x}_v\|\, \Delta u\, \Delta v$ at a point $(u_0, v_0) \in U$, i.e. briefly $\Delta S \approx \|\vec{x}_u \times \vec{x}_v\|\, \Delta u\, \Delta v$



We need to make extra requirements:
- $\vec{x}$ is C^1
- $\vec{x}$ is one-to-one on U
- $\{\vec{x}_u, \vec{x}_v\}$ are linearly independent, or alternatively, $D\vec{x}$ has rank 2

Let's also assume that $U = [a, b] \times [c, d]$, that is, is a rectangle.
In this case, we cut-up U into sub-rectangles then add up each ΔS. This should reasonably approximate the area of the surface $\vec{x}(U)$ in $\mathbb{R}^3$.
We now define the **surface area** of M over a single patch, denoted by $\mathrm{area}(M)$, to be

$$\boxed{\mathrm{area}(M) = \iint_U \sqrt{\det(g_{ij})}\, du\, dv}$$

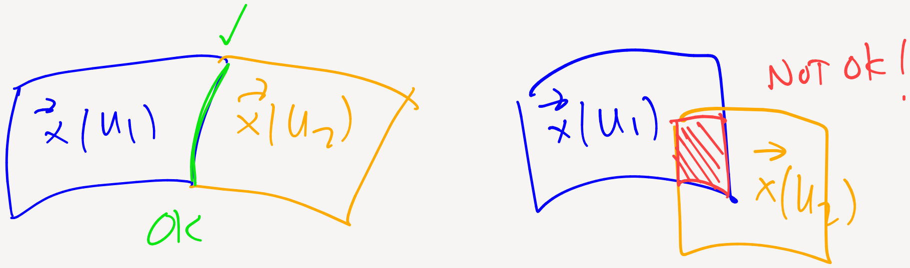


write M as union of patches $\vec{x}(U_i)$ as in step 1.

- **Step 2**

For more general surfaces M, we can cover M with finitely patches of the type in Step 1.

We need to be careful that they only overlap in "one-dimensional sets" (like at points or boundary curves). Then add up the results using the definition of area in Step 1.



- This suggest the following definiton:

Defn 2 Surface Area

Let $M \subset \mathbb{R}^3$ be a smooth surface. Let $\vec{x} : U \subset \mathbb{R}^2 \to M$ be a regular patch. We define the **surface area** of M, denoted by $\mathrm{area}(M)$, to be the number

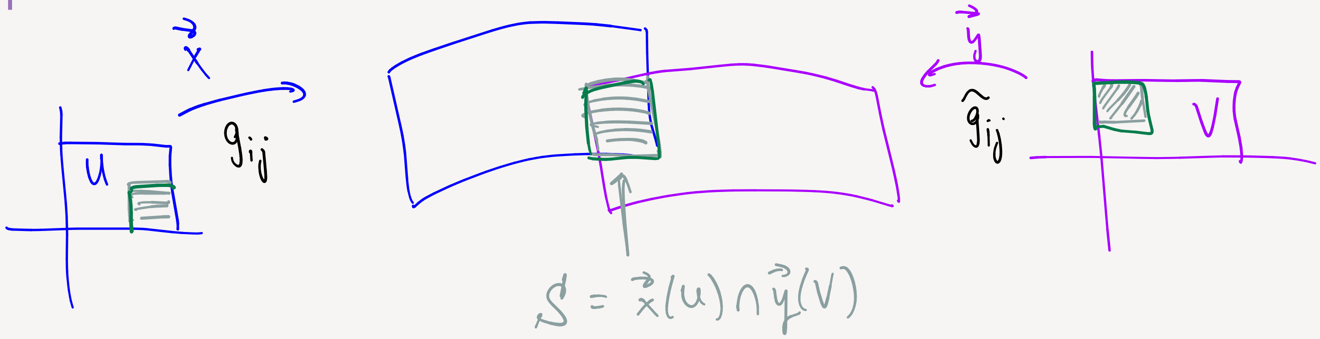
$$\boxed{\mathrm{area}(M) = \iint_U \sqrt{\det(g_{ij})}\, du\, dv} \tag{2}$$

Remarks A few thoughts:

1. Since $\vec{x}$ is C^1, the g_{ij} are continuous $\implies \det(g_{ij})$ is continuous (another fun fact from linear algebra is that the determinant is continuous function) $\implies \sqrt{\det(g_{ij})}$ is continuous $\implies$ the integral exits.
2. A key property that we need to check is that the formula for the area doesn't depend on the parametrization chosen.

   That is, if we have a different regular patch $\vec{y} : V \subset \mathbb{R}^2 \to \mathbb{R}^3$ with $\vec{y}(V) = M = \vec{x}(U)$. Then both integrals agree. This is Lemma 5.8.3 in Bloch's book, which I encourage you to read.

   The reason the surface area is independent of parametrization is essentially due to the "change of variables" formula.



Must check

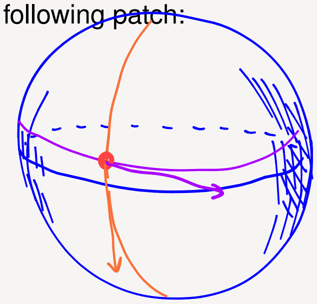
$$\iint_{\vec{x}^{-1}(S)} \sqrt{\det(g_{ij})}\, du\, dv = \iint_{\vec{y}^{-1}(S)} \sqrt{\det(\tilde{g}_{ij})}\, d\tilde{u}\, d\tilde{v}$$

$$\phi : U \subseteq \mathbb{R}^2 \longrightarrow V \subseteq \mathbb{R}^2$$

$$\phi = \vec{y}^{-1} \circ \vec{x}\,\big|_S$$ "change of variables"

• Examples

Act 1 Sphere Let $a > 0$ and $M \subset \mathbb{R}^3$ be the sphere of radius a. Parametrize the sphere using the following patch:



$$\vec{x}(u,v) = \begin{pmatrix} (a\cos v)\cos u \\ (a\cos v)\sin u \\ a\sin v \end{pmatrix} \qquad \text{with} \qquad U = \overset{u}{(-\pi,\pi)} \times \overset{v}{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$$

Compute the **surface area** of the sphere of radius a.

Solution

- $D\vec{x} = [\, \vec{x}_u \mid \vec{x}_v \,] = \begin{bmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \\ \frac{\partial x_3}{\partial u} & \frac{\partial x_3}{\partial v} \end{bmatrix} = \begin{array}{cc} \vec{x}_u & \vec{x}_v \\ \left[\begin{array}{c|c} (a\cos v)(-\sin u) & (-a\sin v)\cos u \\ (a\cos v)(\cos u) & (-a\sin v)\sin u \\ 0 & a\cos v \end{array}\right] \end{array}$

I'll leave it as an exercise to verify that $D\vec{x} \neq \mathbf{0}$ and has rank 2.

- $[\, g_{ij} \,] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \left[\begin{array}{c|c} \vec{x}_u \cdot \vec{x}_u & \vec{x}_u \cdot \vec{x}_v \\ \hline \vec{x}_v \cdot \vec{x}_u & \vec{x}_v \cdot \vec{x}_v \end{array}\right] = \left[\begin{array}{c|c} a^2\cos^2 v & 0 \\ \hline 0 & a^2 \end{array}\right]$

- $g_{11} = \vec{x}_u \cdot \vec{x}_u = (-a\cos v\sin u)^2 + (a\cos v\cos u)^2 + 0^2 = a^2\cos^2 v$

- $g_{22} = \vec{x}_v \cdot \vec{x}_v = (-a\sin v\cos u)^2 + (-a\sin v\sin u)^2 + (a\cos v)^2 = a^2$

- $g_{12} = \vec{x}_u \cdot \vec{x}_v = (+a^2\cos v\sin u\sin v\cos u) + (-a^2\cos v\cos u\sin v\sin u) + 0 = 0$

$\Longrightarrow \sqrt{\det(g_{ij})} = \sqrt{a^4\cos^2 v} = a^2\cos v$

- So

$$\text{area}(M) = \iint_U \sqrt{\det(g_{ij})}\, du\, dv = \int_{-\pi/2}^{\pi/2}\int_{-\pi}^{\pi} a^2\cos v\, du\, dv = 2\pi a^2 \int_{-\pi/2}^{\pi/2} \cos v\, dv$$

$$= \boxed{4\pi a^2 \text{ sq. units}}$$

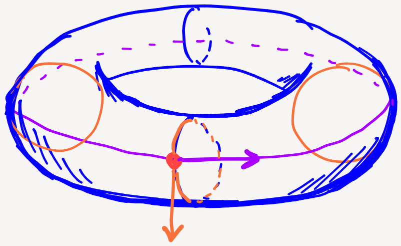
Act 2 Torus Let $R > r > 0$ be constants. Parametrize the torus using the following patch:

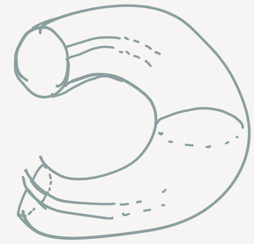
$$\vec{x}(u,v) = \begin{pmatrix} (R + r\cos u)\cos v \\ (R + r\cos u)\sin v \\ r\sin u \end{pmatrix} \qquad \text{with} \qquad U = \overset{u}{(0,2\pi)} \times \overset{v}{(0,2\pi)}$$

Compute the **surface area** of the torus.

Solution

Sketch of M:





- $$D\vec{x} = [\vec{x}_u | \vec{x}_v] = \begin{bmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \\ \frac{\partial x_3}{\partial u} & \frac{\partial x_3}{\partial v} \end{bmatrix} = \begin{array}{cc} \vec{x}_u & \vec{x}_v \end{array}$$
$$\begin{bmatrix} (-r\sin u)\cos v & (R + r\cos u)(-\sin v) \\ (-r\sin u)\sin v & (R + r\cos u)(\cos v) \\ r\cos u & 0 \end{bmatrix}$$

I'll leave it as an exercise to verify that $D\vec{x} \neq \mathbf{0}$ and has rank 2.

- $$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \vec{x}_u \cdot \vec{x}_u & \vec{x}_u \cdot \vec{x}_v \\ \vec{x}_v \cdot \vec{x}_u & \vec{x}_v \cdot \vec{x}_v \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & (R + r\cos u)^2 \end{bmatrix}$$

- $g_{11} = \vec{x}_u \cdot \vec{x}_u = (-r\sin u\cos v)^2 + (-r\sin u\sin v)^2 + (r\cos u)^2 = r^2.$

- $g_{22} = \vec{x}_v \cdot \vec{x}_v = (R + r\cos u)^2\sin^2 v + (R + r\cos u)^2\cos^2 v + 0^2 = (R + r\cos u)^2$

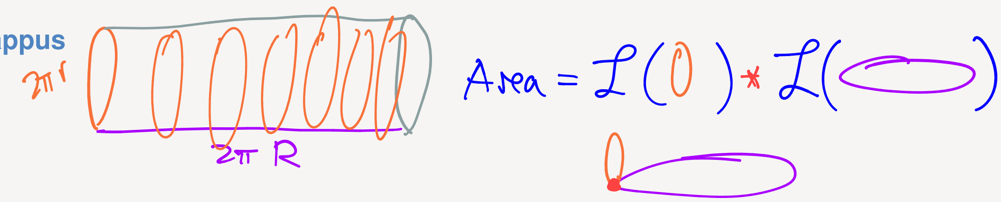
- $g_{12} = \vec{x}_u \cdot \vec{x}_v = (R + r\cos u)(r\sin u\cos v\sin v) + (R + r\cos u)(-r\sin u\sin v\cos v) = 0$

$$\implies \sqrt{\det(g_{ij})} = \sqrt{r^2(R + r\cos u)^2} = r(R + r\cos u)$$

- So
$$\text{area}(M) = \iint_U \sqrt{\det(g_{ij})}\, du\, dv = \underbrace{\int_0^{2\pi} \underbrace{\int_0^{2\pi} r(R + r\cos u)\, du}_{u}\, dv}_{v}$$

$$= 2\pi r \int_0^{2\pi} (R + r\cos u)\, du = 2\pi r \Big[Ru + r\sin u \Big]_0^{2\pi} = 2\pi r [2\pi R] = \boxed{4\pi^2 Rr \text{ units sq}}$$

Bonus: Theorem of Pappus



Bonus: Curious Fact

↳ Hints @ "intrinsic nature" of surface area (hints.... Gauss' Theorema Egregium)

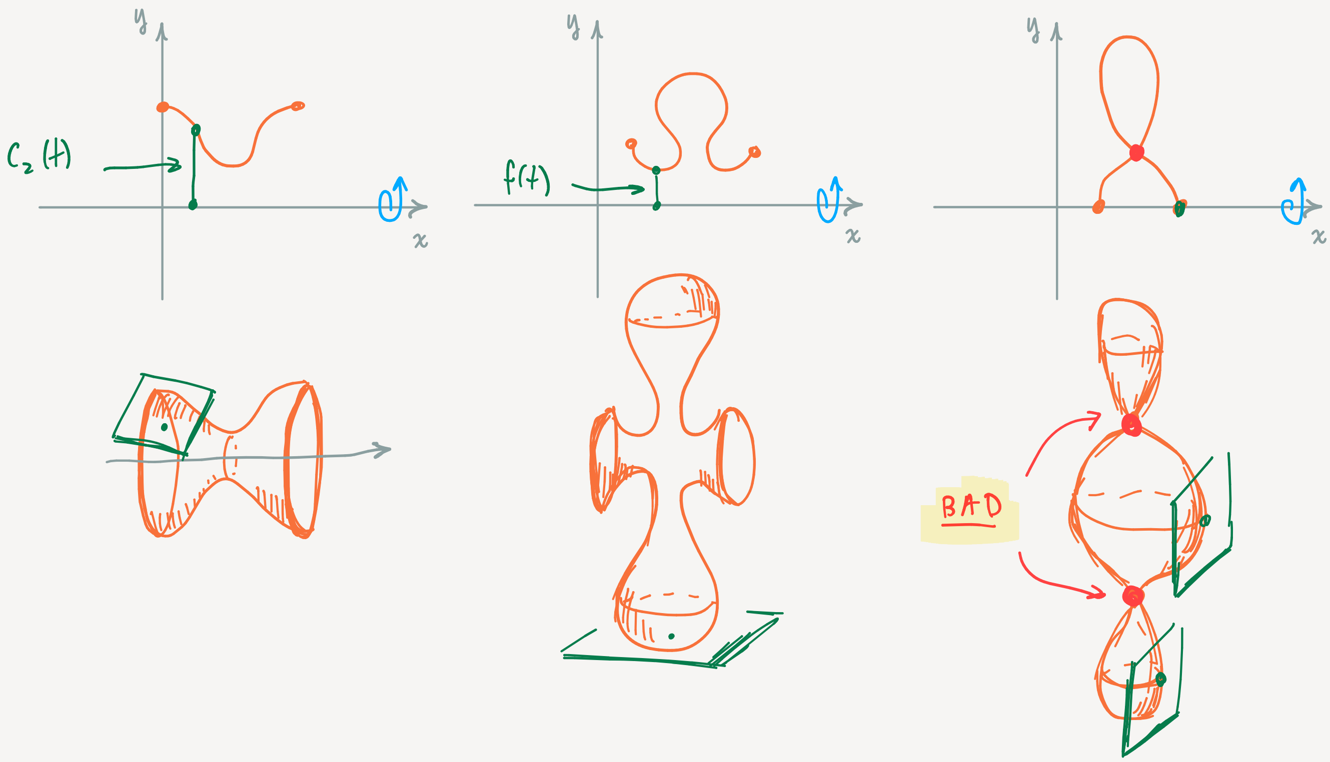
• Surfaces of Revolution

We start with a curve in the plane in parametric form:

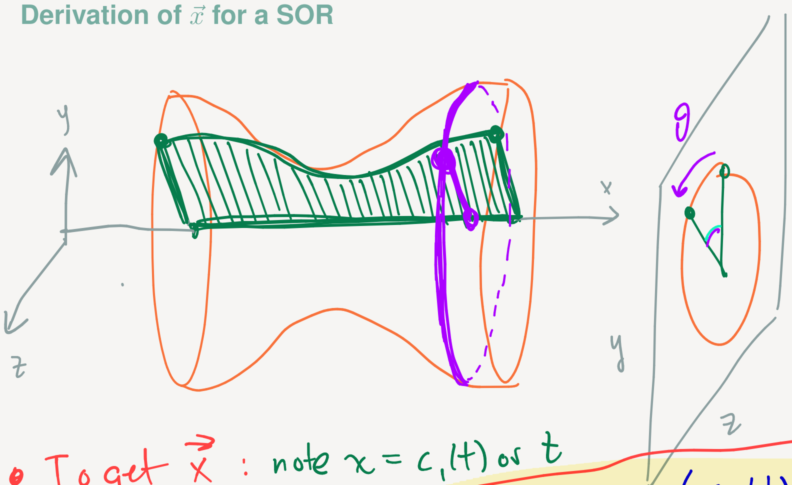
$$\begin{cases} x &= c_1(t) \\ y &= c_2(t) \end{cases} \quad \text{or} \quad \begin{cases} x &= t \\ y &= f(t) \end{cases} \quad \text{for } t \in I$$

AOR : x-axis

Requirements: Assume $c_i(t), f(t)$ are C^1 and also $c_2(t) > 0,\ f(t) > 0$ for $t \in I$.



Derivation of $\vec{x}$ for a SOR



• EQ Circle in yz plane:

$y^2 + z^2 = r^2$

• Parametrize EQ circle in yz plane:

$$\begin{cases} y = r\cos\theta \\ z = r\sin\theta \end{cases} \qquad \begin{cases} y = c_2(t)\cos\theta \\ z = c_2(t)\sin\theta \end{cases}$$

$$\begin{cases} y = f(t)\cos\theta \\ z = f(t)\sin\theta \end{cases}$$

• To get $\vec{x}$: note $x = c_1(t)$ or t

$$\vec{x}(t,\theta) = \begin{pmatrix} c_1(t) \\ c_2(t)\cos\theta \\ c_2(t)\sin\theta \end{pmatrix} = \begin{pmatrix} t \\ f(t)\cos\theta \\ f(t)\sin\theta \end{pmatrix}$$

Defn 3 Surface of Revolution
We can describe the **surface of revolution**, $M = \vec{x}(U)$, using a single patch:

$$\vec{x}(t,\theta) = \begin{pmatrix} c_1(t) \\ c_2(t)\,\cos(\theta) \\ c_2(t)\,\sin(\theta) \end{pmatrix} \quad \text{or} \quad \vec{x}(t,\theta) = \begin{pmatrix} t \\ f(t)\cos\theta \\ f(t)\sin\theta \end{pmatrix} \quad \text{with} \quad U = \begin{array}{ccc} t & & \theta \\ I & \times & (0,2\pi) \end{array}$$

Exercise Describe a **sphere** of radius $a > 0$ and a **torus** of outer radius R and inner radius r $(R > r > 0)$ as surfaces of revolution.

Patch computations

- $\vec{x}(t,\theta) = \begin{pmatrix} t \\ f(t)\cos\theta \\ f(t)\sin\theta \end{pmatrix}$

- $D\vec{x} = \left[\, \vec{x}_t \,\middle|\, \vec{x}_\theta \,\right] = \begin{array}{c} \begin{array}{cc} \vec{x}_t & \vec{x}_\theta \end{array} \\ \left[\begin{array}{c|c} 1 & 0 \\ f'(t)\cos\theta & f(t)\cdot(-\sin\theta) \\ f'(t)\sin\theta & f(t)\cdot(\cos\theta) \end{array}\right] \end{array}$

$f(t) > 0$

Exercise Explain why this proves $D\vec{x} \neq \mathbf{0}$ and that $\operatorname{rank}(D\vec{x}) = 2$ for all points $(t,\theta) \in I$.

- $[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \vec{x}_t \cdot \vec{x}_t & \vec{x}_t \cdot \vec{x}_\theta \\ \vec{x}_\theta \cdot \vec{x}_t & \vec{x}_\theta \cdot \vec{x}_\theta \end{bmatrix} = \left[\begin{array}{c|c} 1 + (f'(t))^2 & 0 \\ \hline 0 & (f(t))^2 \end{array}\right]$

- $g_{11} = \vec{x}_t \cdot \vec{x}_t = 1^2 + (f'(t)\cos\theta)^2 + (f'(t)\sin\theta)^2 = 1 + (f'(t))^2$
- $g_{22} = \vec{x}_\theta \cdot \vec{x}_\theta = 0^2 + (f(t)\cdot -\sin\theta)^2 + (f(t)\cdot\cos\theta)^2 = (f(t))^2$
- $g_{12} = \vec{x}_t \cdot \vec{x}_\theta = 0 + f(t)f'(t)(-\cos\theta\sin\theta) + (f(t)f'(t)\cos\theta\sin\theta) = 0$

$\Longrightarrow \sqrt{\det(g_{ij})} = \sqrt{(1 + [f'(t)]^2)\cdot(f(t))^2} = f(t)\sqrt{1 + [f'(t)]^2}$

- So

$$\operatorname{area}(M) = \iint_U \sqrt{\det(g_{ij})}\,du\,dv = \underbrace{\int_0^{2\pi} \underbrace{\int_I f(t)\sqrt{1 + [f'(t)]^2}\,dt}_{t}\,d\theta}_{\theta}$$

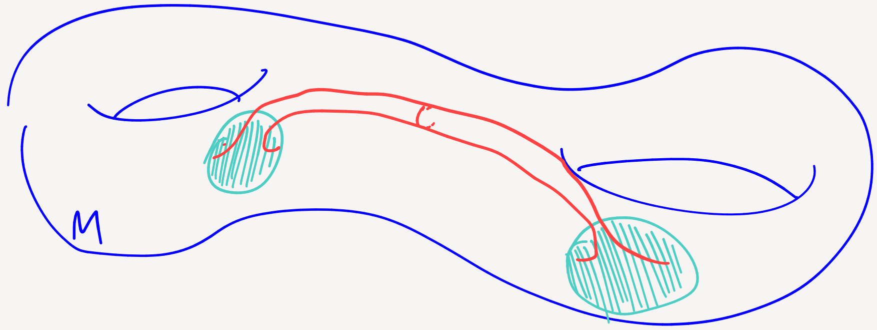
$$= \boxed{2\pi \int_I f(t)\sqrt{1 + [f'(t)]^2}\,dt}$$

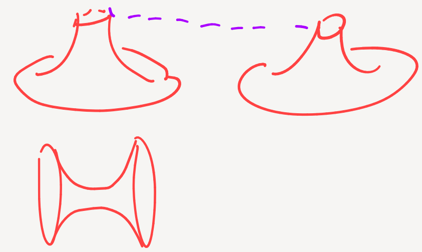
Exercise Do the patch computations for a SOR M parametrized with $\vec{x}(t,\theta) = \langle c_1(t), c_2(t)\cos(\theta), c_2(t)\sin(\theta)\rangle$ and verify that

$$\operatorname{area}(M) = 2\pi \int_I c_2(t)\sqrt{c_1(t)^2 + c_2(t)^2}\,dt.$$

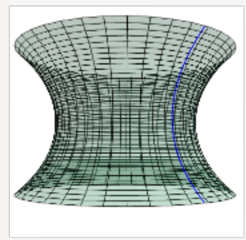
• Wormholes

A **wormhole** to geometers is essentially a piece of a torus that cut and attached back to a surface. These are also called **tunnels**.





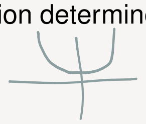
We'll look at a few examples using SageMath: https://jorgemathbasilio.github.io/inspiring_talks/surface_area.html



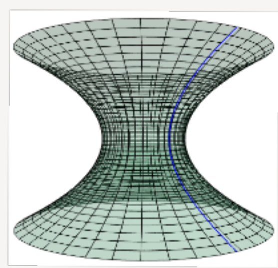
Act 3 Wormhole 1 Consider the curve C in $\mathbb{R}^2$ parametrized by

$$\begin{cases} x &= t \\ y &= \cosh(t) \end{cases} \qquad \text{for } t \in I = (-1,1)$$

Let $M = \vec{x}(U)$ be the surface of revolution determined by C.
Compute the **surface area** of M.



- $f(t) = \cosh(t) = (e^{t} + e^{-t})/2$
- $f'(t) = \sinh(t)$
- $\sqrt{1+[f'(t)]^2} = \sqrt{1+\sinh^2(t)} = \sqrt{\cosh^2(t)} = \cosh(t)$
- $\text{area}(M) = 2\pi \int_I f\sqrt{1+[f']^2}\,dt = 2\pi \int_{-1}^{1} \cosh^2(t)\,dt = 2\pi \int_{-1}^{1} \frac{1+\cosh(2t)}{2}\,dt = \boxed{\pi\left[2+\sinh(2)\right] \text{ unit. sq.}}$



Act 4 Wormhole 2 Consider the curve C in $\mathbb{R}^2$ parametrized by

$$\begin{cases} x &= t \\ y &= (1/2)t^2 + 1 \end{cases} \qquad \text{for } t \in I = (-2,2)$$

Let $M = \vec{x}(U)$ be the surface of revolution determined by C.
Compute the **surface area** of M.

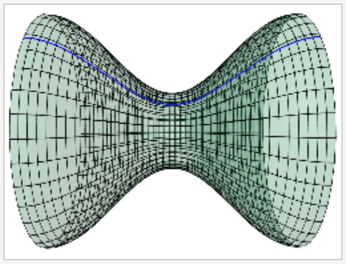
- $f(t) = \frac{1}{2}t^2 + 1$
- $f'(t) = t$
- $\sqrt{1+[f'(t)]^2} = \sqrt{1+t^2}$
- $\text{area}(M) = 2\pi \int_I f\sqrt{1+[f']^2}\,dt = 2\pi \int_{-2}^{2} \left(\frac{1}{2}t^2+1\right)\sqrt{1+t^2}\,dt$

$$= \pi \int_{-2}^{2} t^2\sqrt{1+t^2}\,dt + 2\pi \int_{-2}^{2} \sqrt{1+t^2}\,dt$$

HINT #1 $t = \tan(\theta)$
$dt = \sec^2(\theta)\,d\theta$
$\sqrt{1+t^2} = \sec(\theta)$

HINT #2 IBPs.

Act 5 Wormhole 3 Consider the curve C in $\mathbb{R}^2$ parametrized by



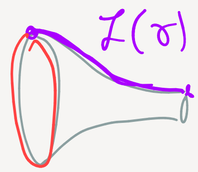
$$\begin{cases} x &= t \\ y &= 2+\cos(t) \end{cases} \qquad \text{for } t \in I = (0, 2\pi)$$

Let $M = \vec{x}(U)$ be the surface of revolution determined by C.
Compute the **surface area** of M.

- $f(t) = 2 + \cos(t)$
- $f'(t) = -\sin(t)$
- $\sqrt{1+[f'(t)]^2} = \sqrt{1+\sin^2(t)}$
- $\text{area}(M) = 2\pi \int_I f\sqrt{1+[f']^2}\,dt = 2\pi \int_0^{2\pi} (2+\cos(t))\sqrt{1+\sin^2(t)}\,dt$

$$= 4\pi \int \sqrt{1+\sin^2(t)}\,dt + 2\pi \int \cos(t)\sqrt{1+\sin^2(t)}\,dt$$

Elliptic Integral
numerical int.

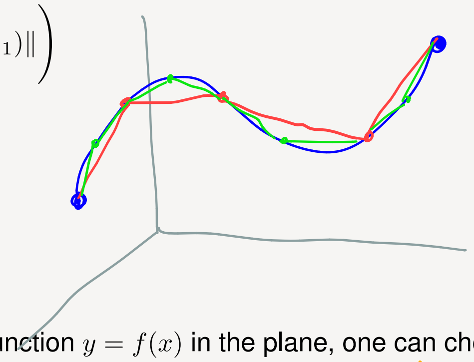


• Arclength

• Length of Curves in M

Given a parametrized curve, $\gamma : I \to \mathbb{R}^3$, we can find the **length** of γ in exactly the same way that you did in calculus:

(def)
$$\text{length}_I(\gamma) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|\right)$$



Using the mean-value theorem, we can then prove the following

Theorem 1 Arclength

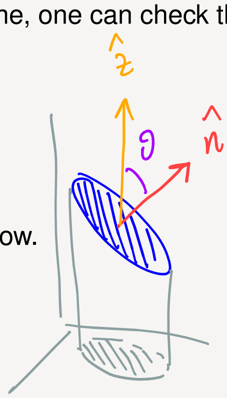
The length of a curve can be computed by a familiar integral:

$$\boxed{\text{length}_I(\gamma) = \int_I \|\gamma'(t)\|\,dt.}$$

Thus, the infinitesimal length is $ds = \|\gamma'(t)\|\,dt$. And recall that for the graph of a function $y = f(x)$ in the plane, one can check that $\|\gamma'(t)\| = \sqrt{1+(f'(t))^2}$.

For surfaces of revolution:

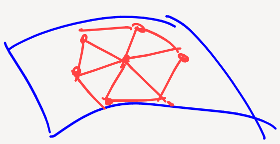
$$dA = (2\pi f(t))\,ds$$



which feels right.

Notice we don't need to really mention that $\gamma \subset M$ if we don't want to. You can and Bloch's book mentions how.

• Some extra remarks

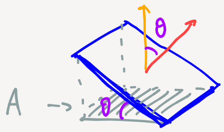


Remarks We close with a few remarks:

- Why can't we do something similar to length with surface area? See "Differential Geometry" by Kreyszig for an example where it can go wrong (§37, pgs. 115–117).
- A neat alternative to the formula for dA we developed is

θ

$$\|\vec{x}_u \times \vec{x}_v\|dudv = \sec(\theta)dudv$$



  where θ is the angle between the normal vector to the surface M in $\mathbb{R}^3$ and the z-axis.
- How to define higher-dimensional versions? To start, one should decide what the k-volume should be in $\mathbb{R}^n$ (here $k \leq n$). Then study how "k-cubes" change under linear transformations so that you can then use derivatives and calculus to set-up the correct integrals.
  A good place to start is by showing: $\text{vol}(Q) = |\det(T)|\,\text{vol}(Q)$ where $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and $\text{vol}(Q) = (b-a)(d-c)(f-e)$ is the volume of a cube in $\mathbb{R}^3$.