MATH 10 - Linear Algebra — Fall 2019

Inner Product Spaces — In-class Assignment #10

§7.1–7.3

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Activity 1: Definitions

Write the precise definitions of the following terms: Let V be an abstract vector space.

(a) an **inner product** $\langle \cdot | \cdot \rangle$ on V. State the 4 properties required carefully.

(b) an **inner product spcae** $(V, \oplus, \odot, \langle \cdot | \cdot \rangle)$.

(c) the **norm** of a vector $\vec{v}$ in an inner product space V.

(d) the **distance** between two vectors $\vec{v}$ and $\vec{w}$ in an inner product space V.

(e) When $\vec{v}$ and $\vec{w}$ are **orthogonal vectors**, when a set $S = \{\vec{v}_1, \ldots, \vec{v}_k\} \subset V$ is an **orthogonal set**, and when S is an **orthogonal basis** for V.

(f) when S is an **orthonormal set** and when S is an **orthonormal basis**.

(g) the **orthogonal complement** $W^\perp$ of a subspace $W \subset V$.

Activity 2: Computation

Let

$$S = \{\, \langle 1, 0, -2, 1 \rangle, \quad \langle -1, 1, 1, -1 \rangle, \quad \langle 0, 1, -1, 1 \rangle, \quad \langle 1, 0, 1, -1 \rangle \,\}$$

(a) Apply the **Gram-Schmidt Algorithm** to S.

(b) Let $\vec{v} = \langle -5, 2, -4, 8 \rangle$. Find the **coordinates** $\langle \vec{v} \rangle_B$ where B is the **output** of (a).

For the rest of this activity, let $W = \text{Span}\left(\{\, \langle 1, 0, -2, 1 \rangle, \langle -1, 1, 1, -1 \rangle \,\}\right)$

(c) Find an **orthonormal basis** for W, and one for $W^\perp$.

Hint: you don't need to do any new computations!

(d) Use (b) to find the **orthogonal decomposition** $\vec{v} = \vec{a} + \vec{b}$ where $\vec{a} \in W$ and $\vec{b} \in W^\perp$.

(e) Find the **projection matrix** $[\text{proj}_W]$.

(f) Check that (e) is correct by multiplying it to $\vec{v}$ from part (b). You should get the $\vec{a}$ from part (d).

(g) Find **projection matrix** $[\text{proj}_W]_B$ where B is the output of part (a).

Hint: this should be surprisingly simple!

Notice that your answer in part (g) should be a **diagonal** matrix. Thus, by our knowledge of eigentheory, we know that we can diagonalize $[\text{proj}_W]$ into $[\text{proj}_W]_B$ using a matrix C.

(h) Find an invertible matrix C so that $[\text{proj}_W]_B = C^{-1} * [\text{proj}_W] * C$

Activity 3: Proofs

Let $(V, \oplus, \odot, \langle \cdot | \cdot \rangle)$ inner product spcae.

(a) **Prove:** Let $W = \text{Span}(\{\vec{v}_1, \ldots, \vec{v}_k\}) \subseteq V$. If $\vec{v} \in V$ is orthogonal to each $\vec{v}_j$, for $1 \leq j \leq k$, then $\vec{v}$ is orthogonal to every vectors in W.

(b) **Pick one and provide a proof.** You may want to prove all of these, as I will put one on the test.

(i) **Prove:** The **Cauchy-Scwharz Inequality**: For all vectors $\vec{u}, \vec{v} \in V$,

$$|\langle \vec{u} \,|\, \vec{v} \rangle| \leq \|\vec{u}\| \cdot \|\vec{v}\|.$$

Hint: See your class notes from §1.3 where we proved it for $\mathbb{R}^n$. Adapt that proof by using notation from V. OR, you can consult our textbook for a different proof.

(ii) **Prove:** The **Triangle Inequality**: For all vectors $\vec{u}, \vec{v} \in V$,

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|.$$

Hint: See your class notes from §1.3 where we proved it for $\mathbb{R}^n$. Adapt that proof by using notation from V. OR, you can consult our textbook for a different proof.

(iii) **Prove:** The **Triangle Inequality (Distance Version)**: For all vectors $\vec{u}, \vec{v}, \vec{w} \in V$,

$$dist(\vec{u}, \vec{v}) \leq dist(\vec{u}, \vec{w}) + dist(\vec{w}, \vec{v}).$$

Hint: See your class notes from §1.3 where we proved it for $\mathbb{R}^n$. Adapt that proof by using notation from V. OR, you can consult our textbook for a different proof.

(iv) **Prove:** The **(Generalized) Pythagorean Theorem**: Let $\vec{u}, \vec{v} \in V$.

Then $\vec{u}$ and $\vec{v}$ are **orthogonal** to each other if and only if

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u} + \vec{v}\|^2.$$

Hint: See your class notes from §1.3 where we proved it for $\mathbb{R}^n$. Adapt that proof by using notation from V. OR, you can consult our textbook for a different proof.