MATH 10 - Linear Algebra — Fall 2019

Ch 2 Linear Transformations — In-class Assignment #6

§2.5, 2.6, 2.7

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10_28_2019

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Activity 1: Definitions

Write the precise definitions of the following terms:

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

(a) the **kernel of** T

(b) the **range of** T

(c) the **nullity of** T

(d) the **rank of** T

(e) when T is **one-to-one**

(f) when T is **onto**

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Write the precise definitions of the following terms:

(g) when T is **invertible**

(h) the **inverse operator of** T, T^{-1}

Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$. Write the precise definitions of the following terms:

(i) when A is **invertible**

(j) the **inverse matrix**, A^{-1}

Activity 2: Easy Computation

Find the **inverse** of the matrix $\begin{bmatrix} 2 & -3 \\ 7 & 4 \end{bmatrix}$. You may use the formula given in class for 2×2 matrices if you wish.

Activity 3: Computation

Let $A = \begin{bmatrix} 5 & 9 & -4 \\ 2 & -1 & 7 \\ -3 & -8 & 6 \end{bmatrix}$.

(a) Take turns around the group to decide what row operations to do next with the goal of finding A^{-1} using GJE.

(b) Use your row operations to factor A into elementary matrices.

(Hint: First find the Elementary Matrices, $E_1, E_2, \ldots, E_k$ based on the EROs. Then Recall that $A = E_1^{-1} * E_2^{-1} * \cdots E_{k-1}^{-1} * E_k^{-1}$)

(c) Use A^{-1} to solve the SOE:

$$\begin{aligned} 5x + 9y - 4z &= 3 \\ 2x - y + 7z &= -4 \\ -3x - 8y + 6z &= 11 \end{aligned}$$

Activity 4: Proof

The purpose of this Exercise is to investigate the **kernel** and/or **range** of the composition of two linear transformations. Suppose that:

$$T_1 : \mathbb{R}^n \to \mathbb{R}^k \quad \text{and} \quad T_2 : \mathbb{R}^k \to \mathbb{R}^m$$

are linear transformations.

(a) Adapt this definition in Activity 1 part (a) to write down the definition of $\ker(T_1)$, $\ker(T_2)$, and $\ker(T_2 \circ T_1)$ as set up above. There should be THREE separate definitions. Make sure that you precisely use the symbols $\mathbb{R}^n$, $\mathbb{R}^k$, $\mathbb{R}^m$, $\vec{0}_n$, $\vec{0}_k$, and $\vec{0}_m$, where appropriate.

(b) Two out of the three subspaces that you defined in (b) are subspaces of the same Euclidean space. Which of the two kernels live in the same Euclidean space?

(c) Use your definitions to prove that $\ker(T_1)$ is a **subset** of $\ker(T_2 \circ T_1)$. That is,

$$\ker(T_1) \subseteq \ker(T_2 \circ T_1).$$

(Hint: This means that you must show that every vector $\vec{v}$ that satisfies the definition of $\ker(T_1)$ also satisfies the definition of $\ker(T_2 \circ T_1)$. Start with a vector $\vec{v}$ belonging to $\ker(T_1)$ and show that it also belongs to $\ker(T_2 \circ T_1)$.)

(d) Use part (c) to prove that if $T_2 \circ T_1$ is one-to-one, then T_1 is also one-to-one.

(e) Write the contrapositive to the statement in part (d).

Activity 5: Some important Theorems

Here we review a few important theorems.

(a) The following theorems are each equivalent statements to T being one-to-one. Fill-in the blank:

T is 1-1 iff [if $T(\vec{v}_1) = T(\vec{v}_2)$ then __________]

T is 1-1 iff $\ker(T) =$ __________

T is 1-1 iff $\text{nullity}(T) =$ __________

T is 1-1 iff $NS([T])$ is __________

T is 1-1 iff $\text{RREF}([T])$ has _____ free-variables

(b) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. That is all the information we are given. Fill-in the blank:

- The rank(T) = ________________ (select ONE: # leading or free variables)
- The nullity(T) = ________________ (select ONE: # leading or free variables)
- If $n < m$, then we know that T cannot be ________________ (select ONE: 1-1 or onto)
- If $n > m$, then we know that T cannot be ________________ (select ONE: 1-1 or onto)

Activity 6: Computation

Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be a linear transformation. Suppose that $A = [T]$ is given by

$$A = [T] = \begin{bmatrix} 3 & 6 & 3 & -2 & -1 \\ -5 & -7 & -11 & 3 & -4 \\ -2 & -3 & -4 & 4 & 3 \\ 2 & 5 & 0 & 7 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} R = \begin{bmatrix} 1 & 0 & 5 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find a **basis** for $\ker(T)$.

(b) Is T **one-to-one**? Why or why not?

(c) Find a **basis** for range(T).

(d) Is T **onto**? Why or why not?

(e) State the **Dimension Theorem**, in general, for any linear transformation.

(f) Verify the Dimension Theorem for the linear transformation above.