Linear Trans. and Determinants — In-class Assignment #8

§3.4–3.8, 5.1–5.3

Dr. Jorge Basilio

gbasilio@pasadena.edu

PASADENA CITY COLLEGE

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Activity 1: Definitions

Write the precise definitions of the following terms: Let $(V, \oplus, \odot)$ be an abstract vector space.

(a) a **subspace** W of V.

(b) a **basis** B of a subspace $W \subset V$.

(c) The **dimension** of a subspace W.

(d) Let V and W be abstract vector spaces. Define a **linear transformation** $T : V \to W$.

(e) a **one-to-one** linear transformation $T : V \to W$.

(f) a **onto** linear transformation $T : V \to W$.

(g) when V and W are **isomorphic** vector spaces.

(h) State using precise notation (as given in class/book) what the **matrix representation** for $T : V \to W$ is given the ordered bases $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ for V and $B' = \{\vec{w}_1, \dots, \vec{w}_m\}$ for W. Make sure you give the overall notation for the matrix and what the columns are.

Activity 2: Computation

Let $S = \{\sin(x)e^x, \cos(x)e^{-x}\}$. Consider the linear operator on $\mathrm{Span}(S)$ defined by $T = 3 \cdot \mathcal{D} + 5 \cdot \mathrm{Id}$. More explicitly: $T(f(x)) = (3 \cdot \mathcal{D} + 5 \cdot \mathrm{Id})(f(x)) = 3 \cdot \mathcal{D}(f(x)) + 5 \cdot \mathrm{Id}(f(x))$, or

$$T(f(x)) = 3f'(x) + 5f(x)$$

(a) Compute: $T(2\sin(x)e^x + 3\cos(x)e^{-x})$.

(b) Use your calculation in part (a), to find the smallest vector space W so that $T(f(x)) \in W$ for all $f(x) \in \mathrm{Span}(S)$. That is, so that $T : W \to W$ is a linear operator on W.

Hint: this should require little to no work!

(c) Find the **kernel** $\ker(T)$.

(d) (Optional) Find the **range** $\mathrm{range}(T)$.

Activity 3: Computation

Let $T : \mathbb{P} \to \mathbb{P}$ be the function:

$$T(p(x)) = x \cdot p(x).$$

Recall that $\dim(\mathbb{P}) = \aleph_0$.

(a) Show that T is a **linear transformation**.

(b) Show that T is **one-to-one**.

(c) Show that T is NOT **onto**.

Activity 4: Computation

Suppose that $T : \mathbb{P}^2 \to \mathbb{P}^3$ is a linear transformation given by

$$T(p(x)) = x \cdot p(x) + p'(x)(x^2 - 1) + p(-2)(x^3 + 1).$$

(a) Which conclusion can we make right away, even without the matrix of T? That T is: not one-to-one? not onto? Explain briefly your answer.

(b) Warm-up: **Compute** $T(3x^2 + 5x - 2)$.

(c) Explain briefly why $T(p(x))$ is in $\mathbb{P}^3$ if $p(x)$ is from $\mathbb{P}^2$.

(d) Verify that T is a **linear transformation**.

Hint: This means you need to verify two properties: additivity and homogeneity.

(e) Now, let $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3\}$ be ordered bases. **Find** $[T]_{B,B'}$.

(f) Check (a) using Encode, Multiply, and Decode.

(g) Use technology, to find the **RREF** of the matrix representation of T found in part (d).

(h) Based on your matrix in the previous part, decide if T is **one-to-one**. Explain your answer.

(i) Based on your matrix in the previous part, decide if T is **onto**. Explain your answer.

(j) Find the **kernel** $\ker(T)$, if possible.

(k) Find the **range** $\text{range}(T)$, if possible.

(l) State the **rank** and **nullity** of T.

(m) State the **Dimension Theorem** for general linear transformations between abstract vector spaces V and W. Verify the dimension theorem holds for T.

Activity 5: Computation

Our goal is to find the determinant of the matrix A below. Please follow the instructions. Note that the result of each part is used in the next part.

$$A = \begin{bmatrix} -2 & -16 & 24 & -1 \\ -9 & 0 & -36 & 15 \\ 7 & 4 & -4 & 6 \\ 5 & 12 & 16 & 2 \end{bmatrix}$$

(a) Divide the third column by -4. How is the determinant of this new matrix related to $\det(A)$?

(b) Divide the second row of the matrix in (a) by 3. How is the determinant of this new matrix related to $\det(A)$?

(c) Produce a leading 1 in row 1, column 1 by exchanging row 1 and row 3, and then column 1 and column 3, in the matrix in (b). How is the determinant of this new matrix related to $\det(A)$?

(d) Turn the other entires of column 1 into zeros. Show all EROs.

(e) Complete the computation of the determinant of A using EROs to obtain an upper-triangular matrix. Show all EROs.

Activity 6: Proofs

Let V and W be abstract vector spaces. Let $T : V \to W$ be a linear transformation. Assume now that V and W are finite-dimensional vector spaces of dimensions n and m, respectively. That is, $\dim(V) = n$ and $\dim(W) = m$.

(a) **Prove:** Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of **linearly independent** vectors from V. If T is one-to-one, then the set $S' = \{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)\}$ in W is also **linearly independent**.

(b) **Prove:** Assume now that $n = m$. If $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a **basis** for V, then $B' = \{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)\}$ is a **basis** for W.

Hint: use part (a) and the 2 for 1 theorem. This should be easy.

(c) **Prove:** If T is **onto**, then $\dim(V) \geq \dim(W)$.

Hint: use the dimension theorem.

(d) **Prove:** If $\dim(V) > \dim(W)$, then T <u>cannot</u> be **one-to-one**.

Hint: use the dimension theorem.

Optional material

You do NOT need to turn these problems in! These are strictly optional and for your entertainment :-)

Activity 7: Proofs

Let V and W be abtract vector spaces. Note: we are not assuming they are finite-dimensional, they can be infinite dimensional here.

We investigate $\mathcal{L}(V, W)$ which is the set of ALL linear transformations $T : V \to W$.
We make $\mathcal{L}(V, W)$ into a vector space $(\mathcal{L}(V, W), \oplus_{\mathcal{L}(V,W)}, \odot_{\mathcal{L}(V,W)})$ using the usual definitions:

- **Vector Addition:** Given $T_1, T_2 \in \mathcal{L}(V, W)$, define $T_1 \oplus_{\mathcal{L}(V,W)} T_2 \in \mathcal{L}(V, W)$ to be the linear transformation:
$$(T_1 \oplus_{\mathcal{L}(V,W)} T_2)(\vec{v}) = T_1(\vec{v}) + T_2(\vec{v}) \qquad \text{(for all } \vec{v} \in V)$$
Simply put: $(T_1 + T_2)(\vec{v}) = T_1(\vec{v}) + T_2(\vec{v})$.
- **Scalar Multiplication:** Given $T \in \mathcal{L}(V, W)$ and $r \in \mathbb{R}$, define $r \odot_{\mathcal{L}(V,W)} T \in \mathcal{L}(V, W)$ to be the linear transformation:
$$(r \odot_{\mathcal{L}(V,W)} T)(\vec{v}) = r \cdot T(\vec{v}) \qquad \text{(for all } \vec{v} \in V)$$
Simply put: $(r \cdot T)(\vec{v}) = r \cdot T(\vec{v})$.

Go to page 270 of our textbook and mentally check that all 10 VSAs are satisfied.
All that I ask is to tell me what the **zero vector** $\vec{\mathbf{0}}_{\mathcal{L}(V,W)}$ and the **additive inverse** of $T \in \mathcal{L}(V, W)$ are.

Activity 8: Proofs

Our goal is to prove the important result:

Theorem 1: Isomorphism Theorem

Assume that V and W be abtract finite-dimensional vector spaces of dimensions n and m, respectively, i.e. $\dim(V) = n$ and $\dim(W) = m$.

Then $\mathcal{L}(V, W)$ is **isomorpthic** to $\mathbb{M}_{m\times n}$, i.e. $\boxed{\mathcal{L}(V, W) \cong \mathbb{M}_{m\times n}}$.

To get started, we choose ordered bases B for V and B' for W, which we may do by the Existence of a basis theorem. Next, we write them as:

$$B = \{\vec{v}_1, \dots, \vec{v}_n\} \quad \text{and} \quad B' = \{\vec{w}_1, \dots, \vec{w}_m\}.$$

Then we define a function $\Phi : \mathcal{L}(V, W) \to \mathbb{M}_{m\times n}$ as follows: Given $T \in \mathcal{L}(V, W)$, we define $\Phi(T) \in \mathbb{M}_{m\times n}$ via

$$\Phi(T) = [T]_{B,B'}.$$

(a) Show that Φ is a **linear transformation**. You may use Problem 41 from §3.6 without proof.

(b) Show that Φ is **one-to-one**. *Hint: If* $T \in \ker(\Phi)$, *then show it is the zero transformation.*

(c) Show that Φ is **onto** by following the following outline:

  (i) Assume $A \in \mathbb{M}_{m\times n}$ is given. Fill-in the blank:
  We need to show there exits $T \in \mathcal{L}(V, W)$ such that ____________________________.

(ii) We construct $T : B \to W$ as follows: define $[T(\vec{v}_i)]_{B'} = A * [\vec{v}_i]_B$ for each basis vector $\vec{v}_i \in B$. Use the matrix notation $A = (a_{ij})$ and your knowledge of matrix multiplication express $A * [\vec{v}_i]_B$. Then DECODE this to express $T(\vec{v}_i)$ using the basis B'.

(iii) Extend this map to a linear transformation on all of V. *Hint: this is a one liner, there's only one obvious way to do this and don't verify it's a LT.*

(iv) Explain why $T : V \to W$ constructed in parts (ii),(iii) is the desired linear transformation needed to prove Φ is onto.