Chapter 1

The Canvas of Linear Algebra:

Euclidean Spaces and Subspaces

1.1 The Main Subject: Euclidean Spaces

Definition: An ***ordered*** $n-$ ***tuple*** or ***vector*** is an ordered list of n real numbers:

$$\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle.$$

Definition: The set of all possible $n-$ tuples is called ***Euclidean*** $n-$ ***space***, denoted by the symbol $\mathbb{R}^n$:

$$\mathbb{R}^n = \{ \vec{v} = \langle v_1, v_2, \ldots, v_n \rangle \mid v_1, v_2, \ldots, v_n \in \mathbb{R} \}.$$

To distinguish real numbers from vectors, we will also refer to real numbers as ***scalars***.

Definition: Two vectors $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ from $\mathbb{R}^n$ are ***equal*** if all of their components are pairwise equal, that is, $u_i = v_i$ for $i = 1..n$. Two vectors from ***different*** Euclidean spaces are ***never*** equal.

Definitions: Each $\mathbb{R}^n$ has a special element called the ***zero vector***, all of whose components are zero: $\vec{0}_n = \langle 0, 0, \ldots, 0 \rangle$.

Every vector $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle \in \mathbb{R}^n$ has its own ***additive inverse***, also known as its ***negative***:

$$-\vec{v} = \langle -v_1, -v_2, \ldots, -v_n \rangle.$$

Vector Arithmetic

Definitions: If $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ are vectors in $\mathbb{R}^n$, we define the ***vector sum***:

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n \rangle,$$

and if $r \in \mathbb{R}$, we define the ***scalar product***:

$$r \cdot \vec{v} = r\vec{v} = \langle rv_1, rv_2, \ldots, rv_n \rangle.$$

We will call the operation of finding the vector sum as ***vector addition***, and the operation of finding a scalar product as ***scalar multiplication***. We can also define ***vector subtraction*** by:

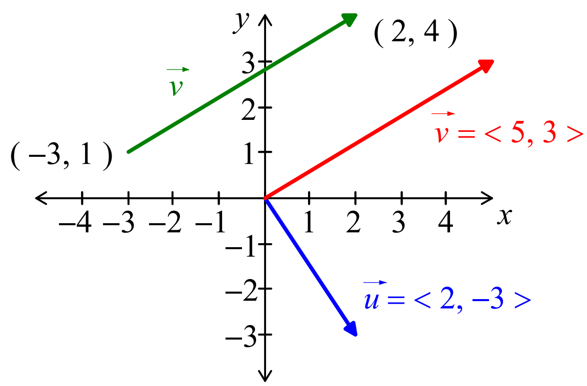
$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}) = \langle u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n \rangle.$$

Theorem: The Multiplicative Property of the Scalar Zero:

Let $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle \in \mathbb{R}^n$. Then:

$$0 \bullet \vec{v} = \vec{0}_n.$$

Visualizing Vectors from $\mathbb{R}^2$



Plotting Vectors in $\mathbb{R}^2$

Translating Vectors in $\mathbb{R}^2$

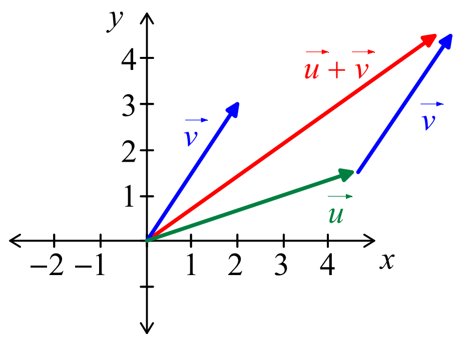
Theorem: Let $\vec{u} = \langle u_1, u_2 \rangle \in \mathbb{R}^2$, and $P(a_1, b_1)$ a point on the Cartesian plane. If $\vec{u}$ is translated to P, then then head of $\vec{u}$ will be located at $Q(a_2, b_2)$, where:

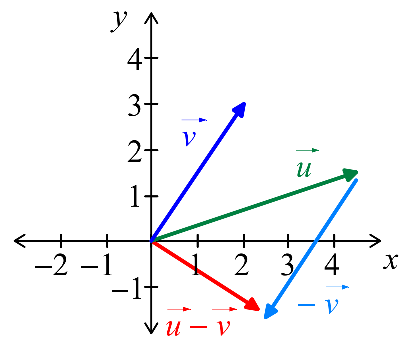
$$a_2 = a_1 + u_1, \text{ and } \ b_2 = b_1 + u_2.$$

Conversely, if $P(a_1, b_1)$ and $Q(a_2, b_2)$ are two points on the Cartesian plane, then the vector $\vec{u} \in \mathbb{R}^2$ from P to Q is:

$$\vec{u} = \overrightarrow{PQ} = \langle a_2 - a_1, b_2 - b_1 \rangle.$$

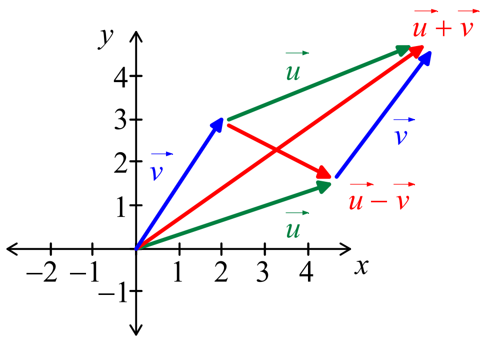
The Geometry of Vector Arithmetic in $\mathbb{R}^2$

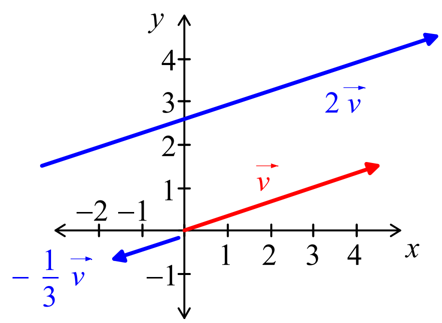




Vector Addition and Subtraction in $\mathbb{R}^2$

The Parallelogram Principle; Scalar Multiplication





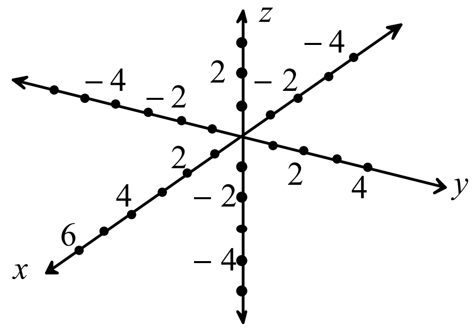
Definition: Axiom for Parallel Vectors:

We say that two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are ***parallel to each other*** if there exists either $a \in \mathbb{R}$ ***or*** $b \in \mathbb{R}$ such that:

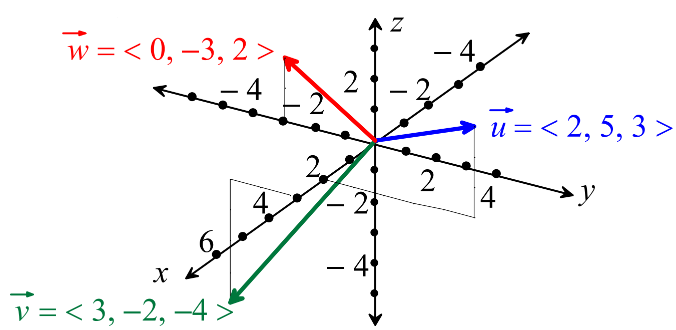
$$\vec{u} = a \cdot \vec{v} \quad \textbf{\textit{or}} \quad \vec{v} = b \cdot \vec{u}.$$

Consequently, this means that $\vec{\mathbf{0}}_n$ is parallel to ***all*** vectors $\vec{v} \in \mathbb{R}^n$, since $\vec{\mathbf{0}}_n = 0 \cdot \vec{v}$.

Visualizing Vectors from $\mathbb{R}^3$



Cartesian Space



Plotting Vectors from $\mathbb{R}^3$

Translating Vectors in $\mathbb{R}^3$

Theorem: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle \in \mathbb{R}^3$, and $P(a_1, b_1, c_1)$ a point in Cartesian space. If $\vec{u}$ is translated to P, then then head of $\vec{u}$ will be located at $Q(a_2, b_2, c_2)$, where:

$$a_2 = a_1 + u_1, \;\; b_2 = b_1 + u_2, \;\;\text{ and }\;\; c_2 = c_1 + u_3.$$

Conversely, let $P(a_1, b_1, c_1)$ and $Q(a_2, b_2, c_2)$ be two points in Cartesian space. Then the vector $\vec{u} \in \mathbb{R}^3$ from P to Q is:

$$\vec{u} = \overrightarrow{PQ} = \langle a_2 - a_1, b_2 - b_1, c_2 - c_1 \rangle.$$

Properties of Vector Arithmetic

Theorem — Properties of Vector Arithmetic:

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$ and r and s are scalars, then the following properties are true:

1. *The Closure Property of Vector Addition*

$$\vec{u} + \vec{v} \text{ is also in } \mathbb{R}^n.$$

2. *The Closure Property of Scalar Multiplication*

$$r\vec{u} \text{ is also in } \mathbb{R}^n.$$

3. *The Commutative Property of Vector Addition*

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}.$$

4. *The Associative Property of Vector Addition*

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}).$$

5. *The Additive Identity Property*

$$\vec{0}_n + \vec{v} = \vec{v} = \vec{v} + \vec{0}_n.$$

6. *The Additive Inverse Property*

$$\vec{v} + (-\vec{v}) = \overrightarrow{0}_n = (-\vec{v}) + \vec{v}.$$

7. *The "Left" Distributive Property*

$$(r + s)\vec{v} = r\vec{v} + s\vec{v}.$$

8. *The "Right" Distributive Property*

$$r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}.$$

9. *The Associative Property of Scalar Multiplication*

$$(rs)\vec{v} = r(s\vec{v}) = s(r\vec{v}).$$

10. *The Unitary Property of Scalar Multiplication*

$$1\vec{v} = \vec{v}.$$

The Length of a Vector:

Definition: Let $\vec{v} = \langle v_1, v_2 \rangle \in \mathbb{R}^2$ and $\vec{w} = \langle w_{1,} w_2, w_3 \rangle \in \mathbb{R}^3$. We define the ***length*** or ***norm*** or ***magnitude*** of these vectors as:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2} \quad \text{and} \quad \|\vec{w}\| = \sqrt{w_1^2 + w_2^2 + w_3^2}\,.$$

We say that $\vec{v}$ is a ***unit vector*** if $\|\vec{v}\| = 1$. Similarly, $\vec{w}$ is a unit vector if $\|\vec{w}\| = 1$.

Theorem: For any scalar $k \in \mathbb{R}$ and vector $\vec{v} \in \mathbb{R}^2$ or $\mathbb{R}^3$: $\| k\vec{v} \| = |k| \| \vec{v} \|$.

Furthermore, $\|\vec{v}\| \geq 0$, and $\|\vec{v}\| = 0$ ***if and only if*** $\vec{v} = \vec{0}_2$ or $\vec{0}_3$. Consequently, if $\vec{v}$ is a non-zero vector, then:

$$\vec{u}_1 = \frac{1}{\| \vec{v} \|}\vec{v} \quad \text{and} \quad \vec{u}_2 = \frac{-1}{\| \vec{v} \|}\vec{v}$$

are ***unit vectors*** parallel to $\vec{v}$.

Linear Combinations

Definition: If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$, and $x_1, x_2, \ldots, x_k \in \mathbb{R}$, then the vector expression:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k$$

is called a ***linear combination*** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ with ***coefficients*** $x_1, x_2, \ldots, x_k$.

Example: If $\vec{u} = \langle 5, -2, 3, -7 \rangle, \vec{v} = \langle -4, 2, 3, -6 \rangle$ and $\vec{w} = \langle 3, 0, -8, 3 \rangle$, then:

$5\vec{u} - 3\vec{v} + 6\vec{w} =$

Example: If $\vec{u} = \langle 5, -2, 4, 6 \rangle$ and $\vec{v} = \langle 1, -5, 2, -3 \rangle$, is it possible to express $\langle 11, 37, -2, 51 \rangle$ as a linear combination of $\vec{u}$ and $\vec{v}$?

The Standard Basis Vectors

Definition: The ***standard basis vectors*** in $\mathbb{R}^n$ are the vectors $\vec{e}_1$, $\vec{e}_2$, …, $\vec{e}_n$ that have 0 in all components except the i^{th} component, which contains 1 :

$$\vec{e}_i = \langle 0, 0, \ldots, 0, 1, 0, \ldots, 0 \rangle.$$

The Standard Basis Vectors of $\mathbb{R}^2$ and $\mathbb{R}^3$

Theorem: Every vector $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{R}^n$ can be expressed ***uniquely*** as a linear combination of the standard basis vectors:

$$\langle x_1, x_2, \ldots, x_n \rangle = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n.$$

The Proof Template

Write the Theorem you are trying to prove, in its entirety.

Paraphrase the Theorem by identifying the ***given conditions*** and the ***conclusion:***

> ***We are given that:***
>
> ***The conclusion we want to reach is:*** or
>
> ***We want to show that:***

Write down the relevant ***Definitions:***

The given conditions mean that:

The conclusions we want to reach mean that:

The notations in the Theorem mean:

Write down any relevant ***Theorems*** that are related to the given conditions or to the conclusion and try to connect everything in your template together in a complete proof.

Use the words and phrases:

let, consider, assume, suppose,

if, if and only if,

thus, therefore, this implies,
we can conclude that,
but, however,

we know that,
our goal is to show that,
according to this Theorem,
we get a contradiction,
let us form the contrapositive,
conversely . . . etc.

Practice!