1.3 The Dot Product and Orthogonality

Definition: If $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ are vectors from $\mathbb{R}^n$, we define their ***dot product***:

$$\vec{u} \circ \vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Example: If $\vec{u} = \langle 4, -3, -6, 5, -2 \rangle$ and $\vec{v} = \langle 3, -5, 4, -7, -1 \rangle$, then:

$\vec{u} \circ \vec{v} =$

Length of a Vector

Definitions: We define the ***length*** or ***norm*** or ***magnitude*** of a vector $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle \in \mathbb{R}^n$ as the non-negative number:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}\,.$$

It follows directly from the definition of the dot product that:

$$\|\vec{v}\|^2 = \vec{v} \circ \vec{v}, \text{ or in other words, } \|\vec{v}\| = \sqrt{\vec{v} \circ \vec{v}}\,.$$

A vector with length 1 is called a ***unit vector***.

Theorem: For any vector $\vec{v} \in \mathbb{R}^n$ and $k \in \mathbb{R}$: $\|k\vec{v}\| = |k|\|\vec{v}\|$.
In particular, if $\vec{v} \neq \vec{0}_n$, then $\vec{u}_1 = \frac{1}{\|\vec{v}\|}\vec{v}$ is the unit vector in the same direction as $\vec{v}$, and $\vec{u}_2 = -\frac{1}{\|\vec{v}\|}\vec{v}$ is the unit vector in the opposite direction as $\vec{v}$. Furthermore:

$$\|\vec{v}\| = 0 \text{ \textit{\textbf{if and only if}} } \vec{v} = \vec{0}_n.$$

Example: The vector $\vec{v} = \langle 3, -2, 5, -4, -8 \rangle$ has length:

$\|\vec{v}\| =$

The two unit vectors parallel to $\vec{v}$ are:

Properties of the Dot Product

Theorem — Properties of the Dot Product:

For any vectors $\vec{u}$, $\vec{v}$, $\vec{w} \in \mathbb{R}^n$ and scalar $k \in \mathbb{R}$, we have:

1. ***The Commutative Property***

$$\vec{u} \circ \vec{v} = \vec{v} \circ \vec{u}.$$

2. ***The Right Distributive Property***

$$\vec{u} \circ (\vec{v} + \vec{w}) = \vec{u} \circ \vec{v} + \vec{u} \circ \vec{w}.$$

3. ***The Left Distributive Property***

$$(\vec{u} + \vec{v}) \circ \vec{w} = \vec{u} \circ \vec{w} + \vec{v} \circ \vec{w}.$$

4. ***The Homogeneity Property***

$$(k \bullet \vec{u}) \circ \vec{v} = k(\vec{u} \circ \vec{v}) = \vec{u} \circ (k \bullet \vec{v}).$$

5. ***The Zero-Vector Property***

$$\vec{u} \circ \vec{0}_n = 0.$$

6. ***The Positivity Property***

$$\text{If } \vec{u} \neq \vec{0}_n, \text{ then } \vec{u} \circ \vec{u} > 0.$$

The last two properties can be combined into one:

7. *The Non-Degeneracy Property*

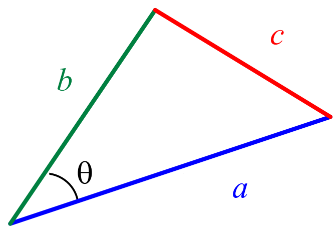
$$\vec{u} \circ \vec{u} > 0 \ \textit{if and only if} \ \vec{u} \neq \overrightarrow{0}_n,$$

$$\text{and } \overrightarrow{0}_n \circ \overrightarrow{0}_n = 0.$$

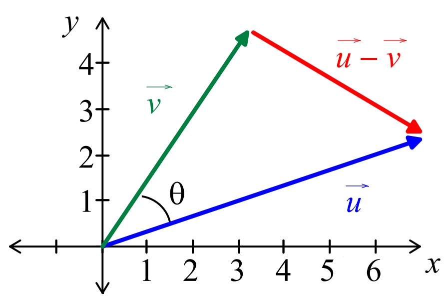
Example: Suppose we are told that $\vec{u}$ and $\vec{v}$ are two vectors from some $\mathbb{R}^n$ (which $\mathbb{R}^n$ is not really important). Suppose we were provided the information that $\|\vec{u}\| = 3$, $\|\vec{v}\| = 7$, and $\vec{u} \circ \vec{v} = 16$. Find $\|4\vec{u} - 9\vec{v}\|$.

A Geometric Formulation for the Dot Product

The Law of Cosines:



$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$



The Triangle Formed by $\vec{v}$, $\vec{u} - \vec{v}$ and $\vec{u}$

Definition/Theorem: If $\vec{u}$ and $\vec{v}$ are ***non-zero*** vectors in $\mathbb{R}^2$, then:

$$\vec{u} \circ \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos(\theta),$$

where θ is the angle formed by the vectors $\vec{u}$ and $\vec{v}$ in standard position. Thus, we can ***compute*** the angle θ between $\vec{u}$ and $\vec{v}$ by:

$$\cos(\theta) = \frac{\vec{u} \circ \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|},$$

where $0 \leq \theta \leq \pi$. We will use the exact same formula for two vectors in $\mathbb{R}^3$.

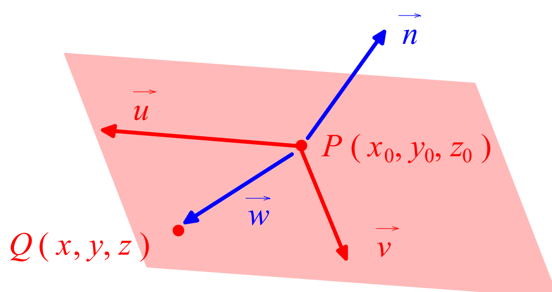
Example: Let us consider the two vectors $\vec{u} = \langle 7, 4 \rangle$ and $\vec{v} = \langle -3, 2 \rangle$.

Orthogonality in $\mathbb{R}^2$ *or* $\mathbb{R}^3$

Definition/Theorem: Two vectors $\vec{u}$ and $\vec{v} \in \mathbb{R}^2$ or $\mathbb{R}^3$ are ***perpendicular*** or ***orthogonal*** to each other ***if and only if*** $\vec{u} \circ \vec{v} = 0$.

Example: $\vec{u} = \langle 4, -2, 3 \rangle$ and $\vec{v} = \langle -3, 5, 7 \rangle$

Revisiting The Cartesian Equation of a Plane



An Arbitrary Plane in Cartesian Space

$$ax + by + cz = d.$$

The Cauchy-Schwarz Inequality

Theorem — The Cauchy-Schwarz Inequality:

For any two vectors $\vec{u}$ and $\vec{v} \in \mathbb{R}^n$: $|\vec{u} \circ \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$.

Proof: We will separate the proof into two cases:

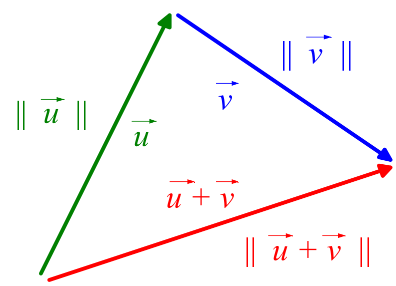
Case 1: Suppose $\vec{u} = \vec{0}_n$ or $\vec{v} = \vec{0}_n$. Then both sides are 0, so the inequality is true.

Case 2: Suppose now that $\vec{u} \neq \vec{0}_n$ and $\vec{v} \neq \vec{0}_n$.

The Triangle Inequality

Theorem — The Triangle Inequality:

For any two vectors $\vec{u}$ and $\vec{v} \in \mathbb{R}^n$: $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$.



The Triangle Inequality: $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Angles and Orthogonality

Definition: If $\vec{u}$, $\vec{v} \in \mathbb{R}^n$ are ***non-zero*** vectors, we define the ***angle*** θ ***between*** $\vec{u}$ ***and*** $\vec{v}$ by:

$$\cos(\theta) = \frac{\vec{u} \circ \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|},$$

where $0 \leq \theta \leq \pi$. Furthermore, we will say that $\vec{u}$ is ***orthogonal*** to $\vec{v}$ if $\vec{u} \circ \vec{v} = 0$.

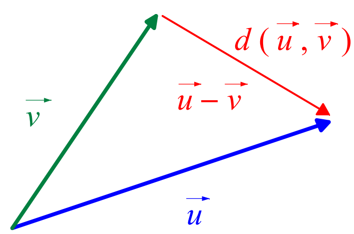
We will ***agree*** that the zero vector $\vec{0}_n$ is orthogonal to ***all*** vectors in $\mathbb{R}^n$.

Example: Find the angle θ between $\vec{u} = \langle 3, -7, 6, -4 \rangle$ and $\vec{v} = \langle 2, 1, -3, -2 \rangle$.

Distance Between Vectors

Definition: If $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ are vectors from $\mathbb{R}^n$, we define the ***distance between*** $\vec{u}$ ***and*** $\vec{v}$ as:

$$\begin{aligned} d(\vec{u}, \vec{v}) &= \|\vec{u} - \vec{v}\| \\ &= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}. \end{aligned}$$



The Distance Between Two Vectors $\vec{u}$ and $\vec{v}$

Example: Let $\vec{u} = \langle 7, 3, -4, -2 \rangle$ and $\vec{v} = \langle -2, 0, 3, -4 \rangle$.

Theorem — Properties of Distances:

Let $\vec{u}$, $\vec{v}$, $\vec{w} \in \mathbb{R}^n$ and $k \in \mathbb{R}$. Then, we have the following properties:

1. *The Symmetric Property for Distances*

$$d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u}).$$

2. *The Homogeneity Property for Distances*

$$d(k\vec{u}, k\vec{v}) = |k| \cdot d(\vec{u}, \vec{v}).$$

3. *The Triangle Inequality for Distances*

$$d(\vec{u}, \vec{w}) \le d(\vec{u}, \vec{v}) + d(\vec{v}, \vec{w}).$$