1.4 Systems of Linear Equations

Question: When is $\vec{b}$ found in $Span\left(\left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right\}\right)$?

Example: Let $\vec{b} = \langle -6,\ 4,\ 8 \rangle$. Is $\vec{b}$ in:

- L : The line $Span\left(\left\{\langle 9, -6, -12 \rangle\right\}\right)$?
- Π_1 : The plane with Cartesian equation $2x - 4y + 5z = 0$?
- Π_2 : The plane with Cartesian equation $2x + 9y - 3z = 0$?

Example: Decide if $\vec{b} = \langle 10, -9, -5, -7 \rangle$ is a member of the Span of the following five vectors from $\mathbb{R}^4$:

$$\langle 3, -4, 1, -6 \rangle, \langle 2, -3, 2, -5 \rangle, \langle 1, 1, -9, 5 \rangle,$$
$$\langle 1, -2, 2, -4 \rangle, \langle 9, -7, -8, -3 \rangle$$

If so, express $\vec{b}$ as a linear combination of these five vectors in the simplest way possible.

Follow up:

Matrices

Definition: A ***matrix*** is a rectangular table of numbers organized into m ***rows*** and n ***columns***. We say that the ***dimension*** of the matrix is $m \times n$, pronounced "m ***by*** n." If A is an $m \times n$ matrix, we call the number or ***entry*** in row i column j as $(A)_{i,j}$ or $a_{i,j}$. Thus we write:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}.$$

In particular, an $m \times 1$ matrix will be called a ***column matrix***, and a $1 \times n$ matrix will be called a ***row matrix***.

If $S = \left\{ \vec{v}_1, \vec{v}_2, .., \vec{v}_n \right\}$ is a set of vectors from $\mathbb{R}^m$, and $\vec{b} \in \mathbb{R}^m$, we can form the $m \times (n+1)$ ***augmented matrix:***

$$A = \left[\vec{v}_1 \;\; \vec{v}_2 \;\; \dots \;\; \vec{v}_n \;\middle|\; \vec{b} \right],$$

where we assemble the vectors in S into ***columns***, and we separate the last column $\vec{b}$ with a dashed line to indicate that it represents the right side of a system of equations. We normally use x_1, x_2, $\dots$, x_n to be the variables ***associated*** to each column.

The Reduced Row Echelon Form

or RREF

Definition: We will say that an $m \times (n+1)$ augmented matrix is in ***row echelon form*** if it satisfies the following conditions:

1. All the rows consisting entirely of zeroes are at the ***bottom*** of the matrix.

2. The first non-zero entry of any row is the number 1. This entry is called a "***leading 1***."

3. If the next row is non-zero, its leading 1 is to the ***right*** of the previous leading 1.

Furthermore, we say that the matrix is in ***reduced row echelon form***, or ***rref***, if the matrix satisfies the additional condition:

4. All the entries ***above*** a leading 1 are also zeroes.

If column j contains a leading 1, we call x_j a ***leading variable***, and column j is a ***leading column.***

Otherwise, we call x_j a ***free variable***.

Conditions 1 and 3 forces all entries ***below*** a leading 1 to be zeroes. Condition 4 forces all entries ***above*** the leading 1 to be zeroes as well.

The vectors $\vec{e}_1,\ \vec{e}_2,\ \ldots,\vec{e}_k$ will appear in the leading columns, in that order, for some $k \leq m$.

Example:

$$A = \left[\begin{array}{cccccc|c} 1 & 0 & -4 & 0 & 9 & 2 & 6 \\ 0 & 1 & 3 & 0 & 6 & -5 & -9 \\ 0 & 0 & 0 & 1 & -2 & 7 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Elementary Row Operations

Definition: An ***elementary row operation*** is any one of the following actions on a matrix:

Type:	Notation:
1. Multiply row i by a nonzero scalar c	$R_i \longrightarrow cR_i$.
2. Exchange row i and row j	$R_i \leftrightarrow R_j$.
3. Add c times row j to row i	$R_i \longrightarrow R_i + cR_j$.

Invariance of Solution Sets

Theorem — The Invariance of Solution Sets:

An elementary row operation ***does not change*** the solution set of an augmented matrix. In other words, if A is an augmented matrix and B is obtained from A using an elementary row operation, then the solution set of the system corresponding to A is exactly the same as the solution set of the system corresponding to B.

The Uniqueness of the RREF

Theorem — The Uniqueness of the Reduced Row Echelon Form:

The reduced row echelon form of a matrix is ***unique***. Thus, if we start with a matrix A and arrive at two matrices H and J using a different sequence of row operations, and both H and J are in rref, then $H = J$.

The Gauss-Jordan Algorithm

The Gauss-Jordan Algorithm

1. Ignore all the leftmost columns that contain only zeros, if there are any.

2. Starting from the top row and going downward, find the first non-zero entry, called the ***pivot***.

3. If the pivot is not in the top row, ***exchange*** the top row with the pivot's row (this is a Type 2 row operation).

4. Produce a leading 1 in the top row by ***dividing*** the entire top row by the pivot (this is a Type 1 row operation). We call this step ***normalizing*** the row.

5. Make the entries below the leading 1 ***all zeroes*** by adding suitable multiples of the top row to each row below it (these are Type 3 row operations).

6. Now, cover the top row, the leading column and all columns to its left, and repeat steps 1 through 5 on the smaller submatrix.

If we were to stop at Step 6, the algorithm above would be called ***Gaussian Elimination***. It results in a matrix in ***row echelon form***. Now we continue from ***right to left,*** working ***upwards*** as we go:

7. Starting at the rightmost leading 1, produce zeroes above the leading 1 by adding suitable multiples of this row to each row above it (again, these are Type 3 row operations).

8. Repeat Step 7 on the next rightmost leading 1, moving leftward until the matrix is in ***reduced row echelon form***.

An Intelligent Modification:

To avoid fractions, produce a leading "1" using a Type 3 operation instead: multiply a row by a suitable constant and add it to another row to get a leading 1.

Success and Failure

$\vec{b}$ is ***not*** in $Span(S)$ if we get a row consisting entirely of zeroes except for a non-zero entry in the rightmost column.

$$\left[\begin{array}{ccccc|c} 1 & -7 & 0 & 6 & 0 & 3 \\ 0 & 0 & 1 & -4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$