1.5 Linear Systems and Linear Independence

Definition:

A linear system is called ***consistent*** if it has ***at least one*** solution.

A linear system is called ***inconsistent*** if it does ***not*** have any solutions.

Theorem: Let $\vec{b} \in \mathbb{R}^m$ and let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors from $\mathbb{R}^m$. Then $\vec{b} \in Span(S)$ ***if and only if*** the system of equations corresponding to the ***augmented*** matrix:

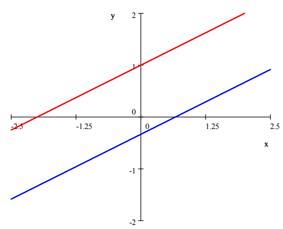
$$A = \left[\, \vec{v}_1 \;\; \vec{v}_2 \;\; \ldots \;\; \vec{v}_n \;\middle|\; \vec{b} \,\right]$$

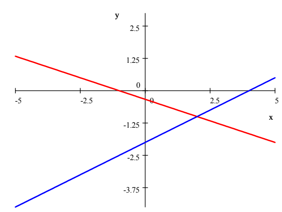
is ***consistent***.

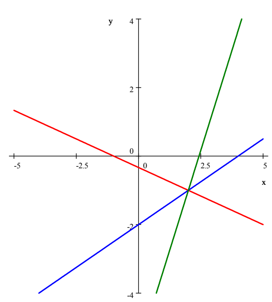
Definition: A linear system with m equations in n variables is called:

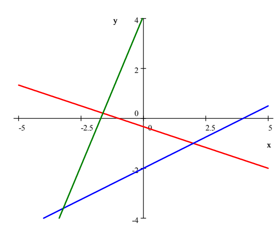
1. ***square*** if $m = n$.
2. ***underdetermined*** if $m < n$.
3. ***overdetermined*** if $m > n$.

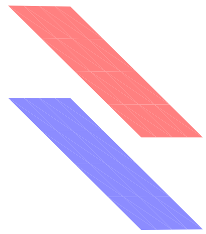
Geometric Interpretation in $\mathbb{R}^2$ *and* $\mathbb{R}^3$

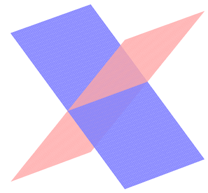


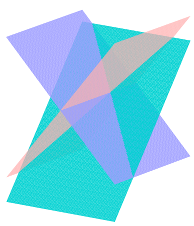


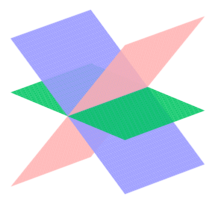


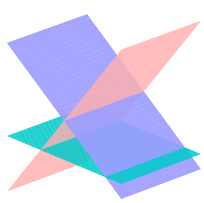


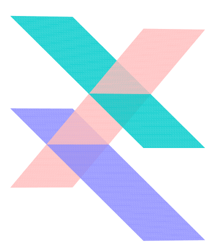












Example: Let us investigate the system:

$$\begin{array}{rcrcrcr} 2x & + & 3y & - & z & = & 5 \\ 5x & + & 4y & - & 3z & = & 7 \\ -7x & + & 7y & + & 6z & = & 10 \end{array}$$

Homogeneous Systems

Definition: A ***homogeneous*** system of m equations in n unknowns is a system of linear equations where the right side of the equations consists entirely of zeros. In other words, the augmented matrix has the form:

$$\left[A \,|\, \vec{0}_m \right],$$

where A is an $m \times n$ matrix. If the right side $\vec{b}$ is not the zero vector, we call the system ***non-homogeneous***.

Clearly, $\vec{x} = \vec{0}_n = \langle 0, 0, \ldots, 0 \rangle$ is a solution to the homogeneous system. We call this the ***trivial solution*** to a homogeneous system, and any other solution is called a ***non-trivial solution***.

When do we get an Infinite Number of Solutions?

Theorem: A homogeneous system has an infinite number of solutions (and hence, non-trivial solutions) ***if and only if*** the rref of A has free variables.

What shape of system always has a free variable?

Theorem: An ***underdetermined homogeneous system*** always has an ***infinite*** number of solutions. In other words, a homogeneous system with ***more variables than equations*** has an infinite number of solutions.

Example:

$$\left[\begin{array}{rrrr|r} 4 & -8 & 3 & 9 & 6 \\ 3 & -6 & -4 & 13 & 17 \\ -2 & 4 & 3 & -9 & -12 \end{array}\right]$$

$$\to \left[\begin{array}{rrrr|r} 1 & -2 & 0 & 3 & 3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Matrix Products

Set-up:

Identify a vector with a column matrix:

$$\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Partition a matrix into ***columns:***

$$A = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix},$$

Definition — Matrix Product:

If $A = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix}$ is an $m \times n$ matrix and $\vec{x} \in \mathbb{R}^n$, we define the ***matrix product*** $A\vec{x}$ to be the linear combination:

$$A\vec{x} = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n.$$

Notice that since each column is an $m \times 1$ matrix, the matrix product is again an $m \times 1$ matrix. Thus, $A\vec{x}$ is a ***linear combination*** of the columns of A with coefficients from $\vec{x}$, and so $A\vec{x} \in \mathbb{R}^m$.

Example:

$$\begin{bmatrix} 7 & -1 & -2 & 6 \\ -2 & 5 & 3 & -4 \\ 8 & 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 3 \\ 5 \end{bmatrix}.$$

Theorem — Properties of Matrix Multiplication:

For all $m \times n$ matrices A, for all $\vec{x}$, $\vec{y} \in \mathbb{R}^n$, and for all $k \in \mathbb{R}$, matrix multiplication enjoys the following properties:

The Additivity Property $\quad A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}.$

The Homogeneity Property $\quad A(k\vec{x}) = k(A\vec{x}).$

The Matrix Product Form of Linear Systems

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{b}.$$

We formed the augmented matrix $\left[\, \vec{v}_1 \; \vec{v}_2 \; \cdots \; \vec{v}_n \mid \vec{b} \,\right]$ and looked at its rref.

Alternative way:

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}.$$

Matrix Equation:

$$A\vec{x} = \vec{b}$$

Rephrase Consistency Requirement for Membership in a Span

Theorem: Suppose that $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors from $\mathbb{R}^m$, and $\vec{b} \in \mathbb{R}^m$. Let us form the $m \times n$ matrix:

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}.$$

Then, $\vec{b} \in \text{Span}(S)$ ***if and only if*** the matrix equation $A\vec{x} = \vec{b}$ is ***consistent***.

Major Concept: Linear Dependence and Independence

Definition: A set of vectors $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ from $\mathbb{R}^m$ is ***linearly dependent*** if we can find a ***non-trivial solution*** $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{R}^n$, where at least one component is ***not zero***, to the vector equation:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}_m.$$

We will call this equation the ***dependence test equation*** for S. An equation of this form where at least one coefficient is ***not zero*** will be referred to as a ***dependence equation***. Thus, for S to be linearly dependent, we must find a non-trivial solution $\vec{x}$ to the homogeneous system:

$$A\vec{x} = \vec{0}_m,$$

where $A = \left[\, \vec{v}_1 \mid \vec{v}_2 \mid \ldots \mid \vec{v}_n \,\right]$ is the matrix with the vectors $\vec{v}_1$, $\vec{v}_2$, …, $\vec{v}_n$ as its ***columns***. This is equivalent to the presence of a ***free variable*** in the rref of the matrix A.

However, if only the trivial solution $\vec{x} = \vec{0}_n$ exists for the dependence test equation, we say that S is ***linearly independent***.

We often drop the adjective “linearly” and simply say that a set S is ***dependent*** or ***independent***.

Example: The standard basis $S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m\}$.

Example: Suppose that $\vec{v}_1 = \langle 4, -5, 3, -2 \rangle$, $\vec{v}_2 = \langle 7, -6, 2, -4 \rangle$ and $\vec{v}_3 = \langle -1, -7, 9, 2 \rangle$.

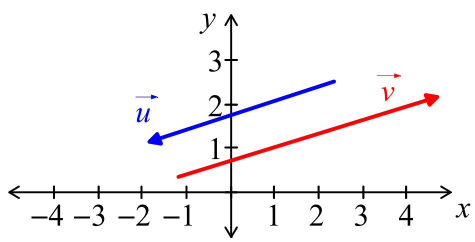
Classifying Small Sets of Vectors

Theorem: Any set $S = \left\{ \overrightarrow{0}_n, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \right\} \subset \mathbb{R}^m$ containing $\overrightarrow{0}_m$ is a ***dependent*** set.

Theorem: A set $S = \{\vec{v}\}$ consisting of a single ***non-zero*** vector $\vec{v} \in \mathbb{R}^m$ is ***independent***.

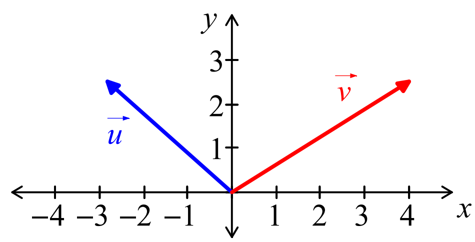
When is $S = \{\vec{u}, \vec{v}\}$ linearly dependent / independent?

Theorem: A set $S = \{\vec{u}, \vec{v}\}$ consisting of ***two*** vectors from $\mathbb{R}^m$ is ***dependent*** if and only if $\vec{u}$ and $\vec{v}$ are ***parallel*** to each other.



Two Dependent

(Parallel) Vectors



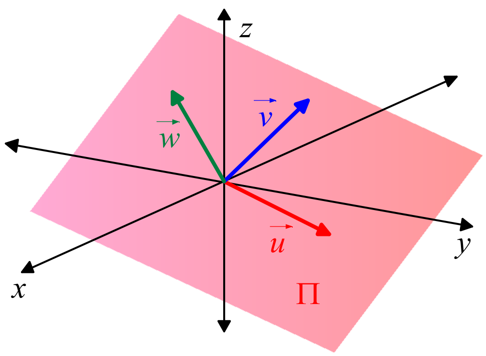
Two Independent

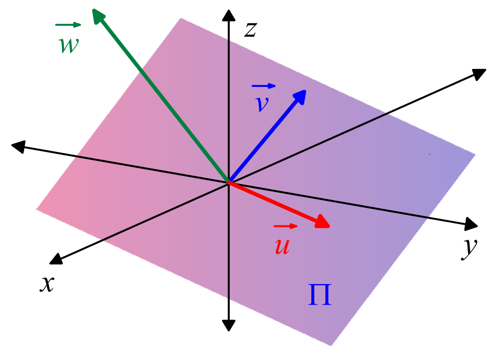
(Non-Parallel) Vectors

Example: $\{\langle 15, -10, 20, -25\rangle, \langle -9, 6, -12, 15\rangle\}$

When is $S = \{\vec{u}, \vec{v}, \vec{w}\}$ linearly dependent / independent?

Theorem: A set $S = \{\vec{u}, \vec{v}, \vec{w}\}$ consisting of ***three*** non-zero vectors from $\mathbb{R}^m$ is ***dependent*** if and only if $\vec{u}$, $\vec{v}$ and $\vec{w}$ are ***coplanar***.





Three Dependent (Non-Parallel) Vectors where $\vec{w} \in Span(\{\vec{u}, \vec{v}\}) = \Pi$

Three Independent (Non-Parallel) Vectors where $\vec{w} \notin Span(\{\vec{u}, \vec{v}\}) = \Pi$

Example:

$$S = \{\langle 2, -3, 4\rangle, \langle 5, 3, -6\rangle, \langle -4, -2, 7\rangle\}.$$

Another Way to Think of Linear Dependence/Independence

Theorem: Suppose that $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a set of non-zero vectors from some $\mathbb{R}^m$, and S contains at least two vectors. Then: S is linearly dependent ***if and only if*** at least one vector $\vec{v}_i$ from S can be expressed as a linear combination of the other vectors in S.

Guaranteed Dependence

If the vectors are from $\mathbb{R}^n$, what is the minimum number of vectors required to produce an underdetermined system?

Theorem: A set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ of m vectors from $\mathbb{R}^n$ is automatically linearly ***dependent*** if $m > n$.

Example:

$$S = \Big\{\Big\langle 5,-3,0,2\Big\rangle, \langle 2,-7,3,-8\rangle, \Big\langle 1,0,-2,4\Big\rangle, \langle -5,1,6,-3\rangle, \langle -2,5,1,6\rangle\Big\}$$