1.7 Subspaces of Euclidean Spaces; Basis and Dimension

Definition: A ***subspace*** W of $\mathbb{R}^n$ is a non-empty subset of vectors of $\mathbb{R}^n$ such that if $\vec{u}, \vec{v} \in W$, and $r \in \mathbb{R}$, then we also have:

$$\vec{u} + \vec{v} \in W \quad \textbf{\textit{and}} \quad r \cdot \vec{v} \in W.$$

We say W is ***closed*** under vector addition and scalar multiplication, and write:

$$W \trianglelefteq \mathbb{R}^n$$

to indicate that W is a subspace of $\mathbb{R}^n$. We call $\mathbb{R}^n$ the ***ambient space*** of W.

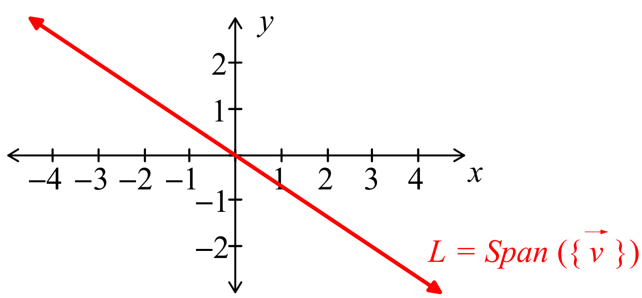
Theorem: The zero vector $\vec{0}_n$ is always a member of any subspace $W \trianglelefteq \mathbb{R}^n$.

Definition/Theorem: For any $\mathbb{R}^n$, there are two ***trivial subspaces:*** (1) the subspace $\left\{\vec{0}_n\right\}$ consisting only of the zero vector, and (2) all of $\mathbb{R}^n$ itself.

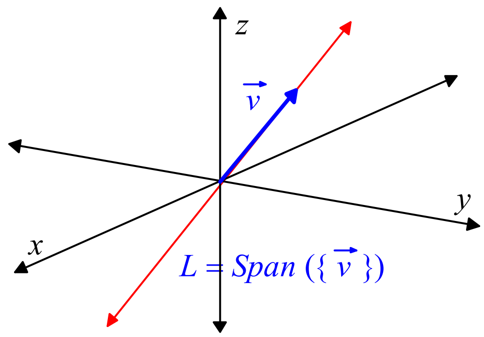
Span(S) as a Subspace

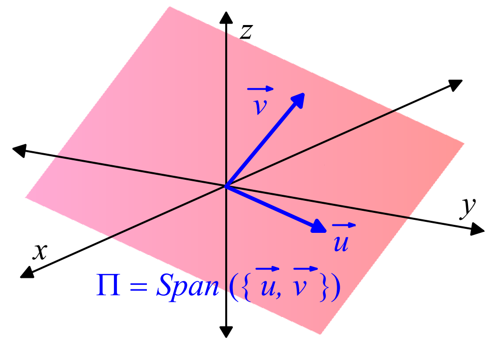
Theorem: If $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is a non-empty set of vectors from $\mathbb{R}^n$, then $W = Span(S)$ is a ***subspace*** of $\mathbb{R}^n$.

Examples:



A Line Through the Origin is a Subspace of $\mathbb{R}^2$





Lines or Planes Through the Origin are Subspaces of $\mathbb{R}^3$

Basis for a Subspace

Definition: A ***basis*** for a non-zero subspace $W \trianglelefteq \mathbb{R}^n$ is a non-empty set of vectors $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ which ***Spans*** W and is also ***linearly independent***.

Example: "Standard basis" $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ for $\mathbb{R}^n$. $\square$

Example:

line through the origin

plane through the origin

A Basis for Span(S)

Theorem: The set $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a ***basis*** for $W = Span(B)$ ***if and only if*** B is linearly independent.

Example: Suppose $S = \{\langle 1, 7, 3, -8, 2\rangle, \langle 4, -2, 5, 3, -4\rangle\}$.

Theorem — The Minimizing Theorem (Basis for Span(S) Version):

Suppose $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ and $W = Span(S)$. If $A = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{bmatrix}$, and R is the rref of A, then the columns of A corresponding to the ***leading columns*** of R form a ***basis*** for W.

Example: Suppose $W = Span(S)$, where:

$$\begin{aligned} S &= \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} \\ &= \{\langle 11, -13, -8, 17\rangle, \langle -4, 7, 3, -6\rangle, \langle 10, -5, -7, 16\rangle\}. \end{aligned}$$

Constructing a Basis for Any Subspace

Theorem — Existence of a Basis Theorem:

If W is any ***non-zero*** subspace of $\mathbb{R}^n$, then ***there exists*** a basis $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ for W. In other words, we can write:

$$W = Span(B) = Span(\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}),$$

where B is a linearly independent set that Spans W. Furthermore, we must have $k \leq n$.

Theorem — The Subspaces of Euclidean 2-Space and 3-Space:

The only subspaces of $\mathbb{R}^2$ are:

(a) the zero subspace $\left\{\vec{0}_2\right\}$,

(b) the ***lines*** through the origin, and

(c) all of $\mathbb{R}^2$.

Similarly, the only subspaces of $\mathbb{R}^3$ are:

(a) the zero subspace $\left\{\vec{0}_3\right\}$,

(b) the ***lines*** through the origin,

(c) the ***planes*** through the origin, and

(d) all of $\mathbb{R}^3$.

The Dimension of a Subspace

Theorem/Definition — The Dimension of a Subspace:

If B and $B^{/}$ are any two bases for the same non-zero subspace $W \trianglelefteq \mathbb{R}^n$, then B and $B^{/}$ contain ***exactly the same number of vectors***. We call this number the ***dimension*** of W, and we write $\boldsymbol{dim}(W) = k$. We also say that W is k-***dimensional***.

We agreed that the trivial subspace $\left\{\vec{0}_n\right\}$ does ***not*** have a basis. By convention, $dim\left(\left\{\vec{0}_n\right\}\right) = 0$.

Conversely, $dim(W)$ is a ***positive integer*** for a ***non-zero*** subspace W.

Example: $dim(\mathbb{R}^n) = ?$

Example:

line through the origin

plane through the origin

Example: $W = Span(S)$, where:

$$S = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$$
$$= \left\{ \begin{array}{c} \langle 11, -13, -8, 17\rangle, \langle -4, 7, 3, -6\rangle, \\ \langle 10, -5, -7, 16\rangle \end{array} \right\}.$$

ambient space?

maximum possible dimension?

basis?

actual dimension?

Example: $W = Span(S)$, where:

$$S = \{\langle 3, -2, 5, 4, 1, -6\rangle, \langle 5, -3, 6, 3, 2, -8\rangle,$$
$$\langle -11, 5, -2, 11, -6, 8\rangle,$$
$$\langle -2, 1, -4, -2, 0, 6\rangle, \langle -4, 1, -1, 7, -1, 6\rangle\}.$$

ambient space?

maximum possible dimension?

$$A = \begin{bmatrix} 3 & 5 & -11 & -2 & -4 \\ -2 & -3 & 5 & 1 & 1 \\ 5 & 6 & -2 & -4 & -1 \\ 4 & 3 & 11 & -2 & 7 \\ 1 & 2 & -6 & 0 & -1 \\ -6 & -8 & 8 & 6 & 6 \end{bmatrix}, \quad \text{with rref:}$$

$$R = \begin{bmatrix} 1 & 0 & 8 & 0 & 5 \\ 0 & 1 & -7 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$