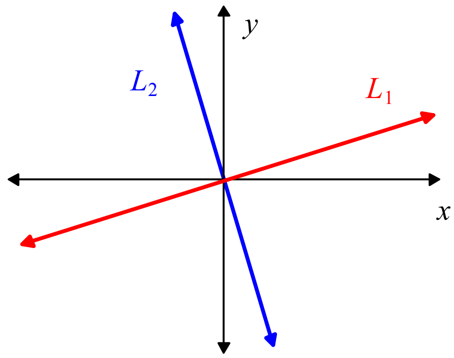
Section 1.9 Orthogonal Complements

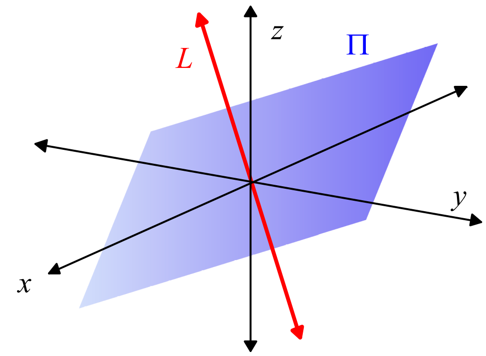
Definition/Theorem: If W is a subspace of $\mathbb{R}^n$, then $W^\perp$ (pronounced *"W perp"*), the *orthogonal complement* of W, defined as:

$$W^\perp = \left\{ \vec{v} \in \mathbb{R}^n \mid \vec{v} \circ \vec{w} = 0 \text{ for } \textbf{\textit{all}}\, \vec{w} \in W \right\}$$

is also a ***subspace*** of $\mathbb{R}^n$.

Examples:





$$L_1 : y = mx,\ m \neq 0, \text{ and } L_2 : y = -\frac{1}{m}x$$

$$\Pi : ax + by + cz = 0, \text{ and } L = Span(\{\langle a,b,c\rangle\})$$

Theorem: If $W = Span(\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}) \trianglelefteq \mathbb{R}^n$, then:

$$W^{\perp} = \left\{ \vec{v} \in \mathbb{R}^n \,|\, \vec{v} \circ \vec{w}_i = 0 \text{ for all } i = 1 \ldots k \right\}.$$

Example: Suppose $S = \{\langle 1, 3, -2, 5\rangle, \langle -2, 5, 7, -8\rangle\}$ and $W = Span(S) \trianglelefteq \mathbb{R}^4$.

Let us find a basis for $W^{\perp}$.

A Dot Product Perspective of Matrix Multiplication

$$A\vec{x} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix} + x_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m,2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{bmatrix}$$

$$\begin{aligned} & x_1 a_{1,1} + x_2 a_{1,2} + \cdots + x_n a_{1,n} \\ = & \langle x_1, x_2, \ldots, x_n \rangle \circ \langle a_{1,1}, a_{1,2}, \ldots a_{1,n} \rangle \\ = & \vec{x} \circ \vec{r}_1, \end{aligned}$$

$$A\vec{x} = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vdots \\ \vec{r}_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \vec{x} \circ \vec{r}_1 \\ \vec{x} \circ \vec{r}_2 \\ \vdots \\ \vdots \\ \vec{x} \circ \vec{r}_m \end{bmatrix} = \begin{bmatrix} \vec{r}_1 \circ \vec{x} \\ \vec{r}_2 \circ \vec{x} \\ \vdots \\ \vdots \\ \vec{r}_m \circ \vec{x} \end{bmatrix}.$$

Theorem: A vector $\vec{x} \in \mathbb{R}^n$ is a solution to $A\vec{x} = \vec{\mathbf{0}}_m$ ***if and only if*** $\vec{x} \circ \vec{r}_i = 0$ for all the rows $\vec{r}_i$ of A. In other words, $\vec{x}$ is in the ***nullspace*** of A ***if and only if*** $\vec{x}$ is ***orthogonal*** to all the ***rows*** of A. Thus:

$$\text{If } W = rowspace(A), \text{ then } W^{\perp} = nullspace(A).$$

$$\text{Similarly, if } U = nullspace(A), \text{ then } U^{\perp} = rowspace(A).$$

Theorem: Suppose $W = Span(\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}) \trianglelefteq \mathbb{R}^n$. If we form the matrix A with ***rows*** $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$, then:

$$W = rowspace(A) \quad \text{and} \quad W^{\perp} = nullspace(A).$$

Thus, the non-zero rows of the rref of A form a basis for W, and we can obtain a basis for $W^{\perp}$ exactly as we would find a basis for $nullspace(A)$ using the rref of A.

Note: This is the ***only*** place in this book where we assemble vectors into the ***rows*** of a matrix. The rest of the time, we will assemble vectors into the columns of a matrix.

Example: $W = Span(S) \trianglelefteq \mathbb{R}^5$, where:

$$S = \{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$$

$$= \left\{ \begin{array}{c} \langle 3, -2, 1, 5, 0\rangle, \langle 5, -3, 2, 6, 1\rangle, \\ \langle -8, 3, -5, 3, -7\rangle, \langle 4, 1, 0, -2, 3\rangle \end{array} \right\}.$$

Theorem — Properties of Orthogonal Complements:

For any subspace $W \trianglelefteq \mathbb{R}^n$:

a) $W \cap W^{\perp} = \left\{ \vec{0}_n \right\}$

b) $(W^{\perp})^{\perp} = W$.

Thus, we can say that W and $W^{\perp}$ are orthogonal complements of ***each other***, or that W and $W^{\perp}$ form an ***orthogonal pair*** of subspaces.

Theorem — The Dimension Theorem for Orthogonal Complements:

If W is a subspace of $\mathbb{R}^n$ with orthogonal complement $W^{\perp}$, then: $dim(W) + dim(W^{\perp}) = n$.

Example: Suppose that:

$$\begin{aligned} S &= \{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\} \\ &= \left\{ \begin{array}{c} \langle 15, -10, 5, 25, 0\rangle, \langle -9, 6, -3, -15, 0\rangle, \\ \langle 1, -2, -5, 11, 8\rangle, \langle 5, -3, 3, 6, -2\rangle \end{array} \right\} \\ &\subset \mathbb{R}^5, \end{aligned}$$

and $W = Span(S)$.

Using dim(W) to Find Other Bases for W

Theorem — The "Two for the Price of One" or "Two-for-One" Theorem:

Suppose W is a subspace of $\mathbb{R}^n$, and $dim(W) = k$. Let $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ be any subset of k vectors from W. Then: B is a basis for W ***if and only if*** either B is linearly independent ***or*** B Spans W. In other words, it is necessary and sufficient to check B for ***only one condition*** without checking the other, if B already contains the ***correct number of vectors***.

easier to ***check linear independence*** rather than Spanning,

Example: In the previous Example:

$$S = \{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$$

$$= \left\{ \begin{array}{c} \langle 15, -10, 5, 25, 0 \rangle, \langle -9, 6, -3, -15, 0 \rangle, \\ \langle 1, -2, -5, 11, 8 \rangle, \langle 5, -3, 3, 6, -2 \rangle \end{array} \right\}.$$

We found that $\dim(W) = 2$, and a basis for W is the set:

$$B = \{\langle 1, 0, 3, -3, -4 \rangle, \langle 0, 1, 4, -7, -6 \rangle\}.$$

Example: Suppose we have $W = Span(S) \trianglelefteq \mathbb{R}^5$, where:

$$S = \{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4, \vec{w}_5\}$$

$$= \left\{ \begin{array}{c} \langle 3, -2, 4, 1, -3\rangle, \langle 1, -1, 2, -1, 0\rangle, \\ \langle 7, -4, 8, 5, -9\rangle, \\ \langle 1, 2, -1, 0, 3\rangle, \langle 5, -1, 2, 11, -12\rangle \end{array} \right\}.$$

$$A = \begin{bmatrix} 3 & -2 & 4 & 1 & -3 \\ 1 & -1 & 2 & -1 & 0 \\ 7 & -4 & 8 & 5 & -9 \\ 1 & 2 & -1 & 0 & 3 \\ 5 & -1 & 2 & 11 & -12 \end{bmatrix},$$

with rref:

$$R = \begin{bmatrix} 1 & 0 & 0 & 3 & -3 \\ 0 & 1 & 0 & -\frac{10}{3} & 5 \\ 0 & 0 & 1 & -\frac{11}{3} & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$A^\top = \begin{bmatrix} 3 & 1 & 7 & 1 & 5 \\ -2 & -1 & -4 & 2 & -1 \\ 4 & 2 & 8 & -1 & 2 \\ 1 & -1 & 5 & 0 & 11 \\ -3 & 0 & -9 & 3 & -12 \end{bmatrix}$$

with rref:

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$