2.1 Mapping Spaces: Introduction to Linear Transformations

Definition: Let X and Y be any two sets. A ***function*** $f : X \to Y$ is a rule (or a recipe, or a formula) that receives as its input an element x of X, and assigns to x as its output a ***unique*** element y of Y.

We write $y = f(x)$, as usual, and also call y the ***image*** of x under f.

We call X the ***domain*** of f, and call Y the ***codomain*** of f. We can also call f a ***map*** (because it tells us where to go), and say that f ***maps*** X ***into*** Y.

Linear Transformations

Definition: A ***linear transformation*** $T : \mathbb{R}^n \to \mathbb{R}^m$ is a function that satisfies:

For all $\vec{u}, \vec{v} \in \mathbb{R}^n$, and for all $k \in \mathbb{R}$:

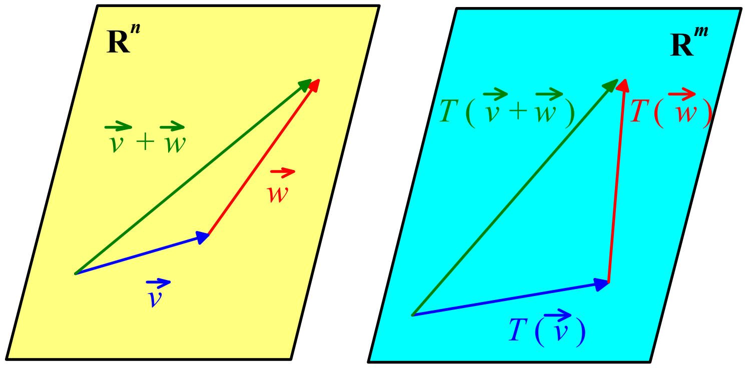
The Additivity Property:

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

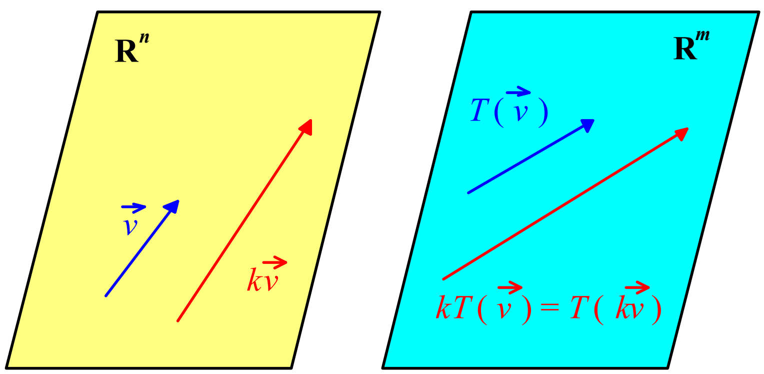
The Homogeneity Property:

$$T(k \cdot \vec{u}) = k \cdot T(\vec{u})$$

In the special case when $T : \mathbb{R}^n \to \mathbb{R}^n$, that is, the domain is the same space as the codomain, we call T a ***linear operator***.



The Additivity Property



The Homogeneity Property

Theorem: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation ***if and only if*** we can find an $m \times n$ matrix A so that the action of T can be performed by matrix multiplication:

$$T(\vec{x}) = A\vec{x}$$

where we view $\vec{x} \in \mathbb{R}^n$ as an $n \times 1$ matrix. We refer to A as the ***standard matrix*** of T, and we write:

$$[T] = A = \left[\, T(\vec{e}_1) \mid T(\vec{e}_2) \mid \dots \mid T(\vec{e}_n) \,\right].$$

In particular, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an ***operator***, $[T]$ is an $n \times n$ or ***square*** matrix.

Some Basic Examples

Example: The zero transformation of $\mathbb{R}^n$ *into* $\mathbb{R}^m$:

$$Z_{n,m} : \mathbb{R}^n \to \mathbb{R}^m, \quad \text{given by}$$
$$Z_{n,m}(\vec{x}) = \vec{0}_m \qquad \text{for all } \vec{x} \in \mathbb{R}^n,$$

$$[Z_{n,m}] = 0_{m \times n} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

For all $\vec{x} \in \mathbb{R}^n$:

$$0_{m \times n}\vec{x} = \vec{0}_m.$$

Example: *The Identity Operator on* $\mathbb{R}^n$:

$$I_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n \quad \text{defined by}$$

$$I_{\mathbb{R}^n}(\vec{x}) = \vec{x} \qquad \text{for all } x \in \mathbb{R}^n$$

The Identity matrix I_n :

$$[I_{\mathbb{R}^n}] = I_n = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$I_n \vec{x} = \vec{x}.$$

Example: ***The Scaling Operators:*** For any $k \in \mathbb{R}$:

$$S_k : \mathbb{R}^n \to \mathbb{R}^n,$$

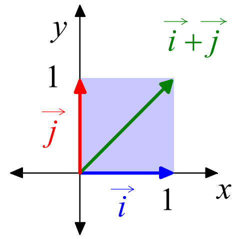
$$S_k(\vec{x}) = k\vec{x} \qquad \text{for all } \vec{x} \in \mathbb{R}^n$$

Elementary Matrices

Definition: An $n \times n$ matrix E is called an ***elementary matrix*** if it is obtained by performing a ***single*** elementary row operation on the identity matrix I_n.

Type:	*Notation:*
1. Multiply row i by $c \neq 0$	$R_i \longrightarrow cR_i$
2. Exchange row i and row j	$R_i \leftrightarrow R_j$
3. Add c times row j to row i	$R_i \longrightarrow R_i + cR_j$

The Basic Box



The Basic Box

Made of the 3 vectors:

$$\left\{\vec{i},\ \vec{j},\ \vec{i}+\vec{j}\right\}$$

Horizontal and Vertical Dilations and Contractions

A 2×2 Type 1 elementary matrix has the form:

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$$

Horizontal Effect

$$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$$

Vertical Effect

Shear Operators

A 2×2 Type 3 elementary matrix has the form:

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$$

A Horizontal Shearing Operator

$$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$$

A Vertical Shearing Operator