2.3 Operations on Linear Transformations and Matrices

Definitions: If $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations, and $k \in \mathbb{R}$, then we can define:

$$T_1 + T_2 : \mathbb{R}^n \to \mathbb{R}^m,$$

$$T_1 - T_2 : \mathbb{R}^n \to \mathbb{R}^m, \text{ and}$$

$$kT_1 : \mathbb{R}^n \to \mathbb{R}^m,$$

as linear transformations, with actions given, respectively, by:

$$(T_1 + T_2)(\vec{v}) = T_1(\vec{v}) + T_2(\vec{v}),$$

$$(T_1 - T_2)(\vec{v}) = T_1(\vec{v}) - T_2(\vec{v}), \text{ and}$$

$$(kT_1)(\vec{v}) = kT_1(\vec{v}).$$

Arithmetic of Matrices

Definitions: If A and B are both $m \times n$ matrices, and k is any scalar, then we can define:

$$A + B,\ A - B,\ \text{and } kA$$

as $m \times n$ matrices with entries given by:

$$(A+B)_{i,j} = (A)_{i,j} + (B)_{i,j},$$
$$(A-B)_{i,j} = (A)_{i,j} - (B)_{i,j}, \quad \text{and}$$
$$(kA)_{i,j} = k(A)_{i,j}.$$

We call these the sum and difference of A and B, and the scalar multiple of A by k.
In particular, we can define the ***negative*** of a matrix, $-A$, to be:

$$-A = (-1)A$$

with the property that:

$$A + (-A) = (-A) + A = 0_{m \times n}$$

Connection Between Linear Transformations and Matrices

Theorem: If $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations, with matrices $[T_1]$ and $[T_2]$ respectively, and k is any scalar, then for any $\vec{v} \in \mathbb{R}^n$:

$$(T_1 + T_2)(\vec{v}) = ([T_1] + [T_2])\vec{v}$$

$$(T_1 - T_2)(\vec{v}) = ([T_1] - [T_2])\vec{v}$$

$$(kT_1)(\vec{v}) = (k[T_1])\vec{v}$$

Consequently, $T_1 + T_2$, $T_1 - T_2$ and kT_1 are linear transformations with matrices given by, respectively:

$$[T_1 + T_2] = [T_1] + [T_2],$$

$$[T_1 - T_2] = [T_1] - [T_2], \text{ and}$$

$$[kT_1] = k[T_1]$$

Compositions of Linear Transformations

Definition/Theorem: If $T_1 : \mathbb{R}^n \to \mathbb{R}^k$ and $T_2 : \mathbb{R}^k \to \mathbb{R}^m$ are linear transformations, then we can define their ***composition***:

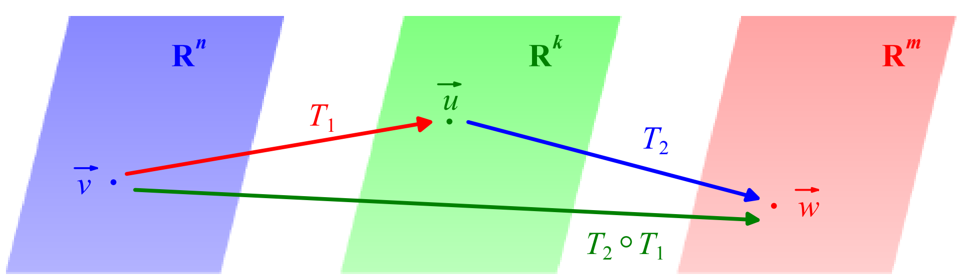
$$T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^m,$$

as a linear transformation, with action given as follows:

Suppose $\vec{u} \in \mathbb{R}^n$, $T_1(\vec{u}) = \vec{v} \in \mathbb{R}^k$, and $T_2(\vec{v}) = \vec{w} \in \mathbb{R}^m$. Then:

$$(T_2 \circ T_1)(\vec{u}) = T_2(T_1(\vec{u})) = T_2(\vec{v}) = \vec{w}.$$

$$\mathbb{R}^n \overset{T_1}{\to} \mathbb{R}^k \overset{T_2}{\to} \mathbb{R}^m$$
$$\searrow \quad T_2 \circ T_1 \quad \nearrow$$



General Matrix Products

Definition — Matrix Product:

If A is an $m \times k$ matrix, and B is a $k \times n$, then we can construct the $m \times n$ matrix AB, where:

$$\text{column } i \text{ of } AB = A \times (\text{column } i \text{ of } B)$$

In other words, if we write B in terms of its columns as:

$$B = \left[\vec{c}_1 \mid \vec{c}_2 \mid \cdots \mid \vec{c}_n \right]$$

then:

$$AB = \left[A\vec{c}_1 \mid A\vec{c}_2 \mid \cdots \mid A\vec{c}_n \right]$$

Linear Combinations of Linear Transformations and Matrices

$$(c_1 T_1 + c_2 T_2 + \cdots + c_k T_k)(\vec{v})$$
$$= c_1 T_1(\vec{v}) + c_2 T_2(\vec{v}) + \cdots + c_k T_k(\vec{v})$$

$$c_1 A_1 + c_2 A_2 + \cdots + c_k A_k.$$

$$[c_1 T_1 + c_2 T_2 + \cdots + c_k T_k]$$
$$= c_1 [T_1] + c_2 [T_2] + \cdots + c_k [T_k]$$