2.5 One-to-One Transformations and Onto Transformations

The Kernel and Range of a Linear Transformation

Definition: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, we define the ***kernel*** of T as the set:

$$ker(T) = \left\{ \vec{v} \in \mathbb{R}^n \,\middle|\, T(\vec{v}) = \vec{0}_m \right\} \subset \mathbb{R}^n.$$

Similarly, we define the ***range*** of T as the set:

$$range(T) = \left\{ \vec{w} \in \mathbb{R}^m \,\middle|\, \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in \mathbb{R}^n \right\} \subset \mathbb{R}^m.$$

We emphasize that $ker(T)$ is from $\mathbb{R}^n$, and $range(T)$ is from $\mathbb{R}^m$.

Theorem: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a ***linear transformation***, then:

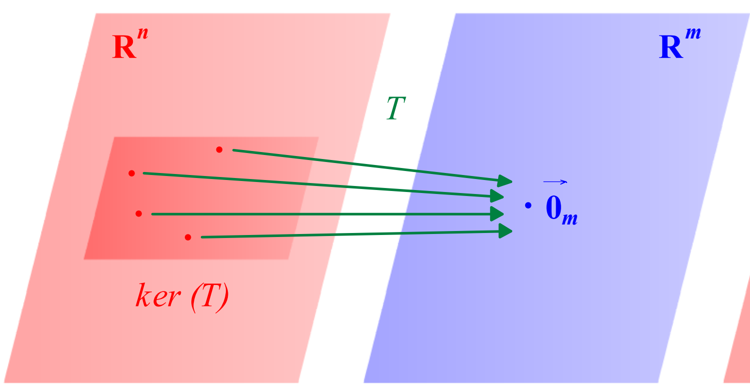
$$ker(T) = nullspace([T]) \trianglelefteq \mathbb{R}^n, \text{ and}$$

$$range(T) = colspace([T]) \trianglelefteq \mathbb{R}^m.$$

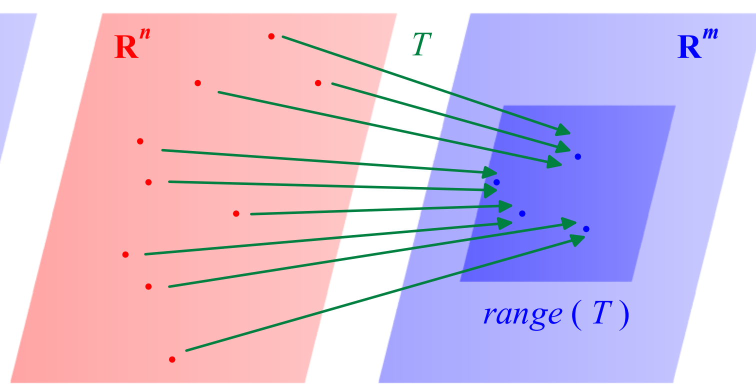
We call the dimension of $ker(T)$ the ***nullity*** of T, written $nullity(T)$. Similarly, we call the dimension of $range(T)$ the ***rank*** of T, written $rank(T)$. Thus:

$$nullity(T) = dim(nullspace([T])) = nullity([T]), \text{ and}$$

$$rank(T) = dim(colspace([T])) = rank([T]).$$



The Kernel of T



The Range of T

The Dimension Theorem for Linear Transformations

Theorem: Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then:

$$rank(T) + nullity(T) = n$$

One-to-One Transformations

Definition: We say that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** or ***injective*** if the image of two different vectors from the domain are different vectors of the codomain:

$$\text{If } \vec{v}_1 \neq \vec{v}_2 \ \text{ then } \ T(\vec{v}_1) \neq T(\vec{v}_2).$$

We also say that T is an ***injection*** or an ***embedding***.

Theorem: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** ***if and only if*** the only way two vectors from the domain have the same image in the codomain is for them to be the same vector to begin with:

$$\text{If } \ T(\vec{v}_1) = T(\vec{v}_2) \ \text{ then } \ \vec{v}_1 = \vec{v}_2.$$

In other words, the ***only solution*** to $T(\vec{v}_1) = T(\vec{v}_2)$ is $\vec{v}_1 = \vec{v}_2$.

Theorem — The Kernel Test for Injectivity:

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** ***if and only if:***

$$ker(T) = \left\{\vec{0}_n\right\}.$$

($\Rightarrow$) We are given that T is one-to-one.

We must show that $ker(T) = \left\{\vec{0}_n\right\}$.

Suppose $\vec{v} \in ker(T)$.

($\Leftarrow$) We are given that $ker(T) = \left\{\vec{0}_n\right\}$.

We must show that T is one-to-one.

So suppose $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$, and $T(\vec{v}_1) = T(\vec{v}_2)$.

Example: Suppose $T_1, T_2 : \mathbb{R}^3 \to \mathbb{R}^4$ are given by the following matrices with the corresponding rrefs. Describe the kernel of each, decide if either is one-to-one, and verify the Dimension Theorem for both.

$$[T_1] = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 9 \\ 5 & -15 & 4 \\ -3 & 9 & -7 \end{bmatrix},$$

$$\text{with rref } R_1 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$[T_2] = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -6 & 9 \\ 5 & -15 & 4 \\ -3 & 9 & -7 \end{bmatrix},$$

$$\text{with rref}\quad R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Describe the kernel and range of each, decide if either is one-to-one, and verify the Dimension Theorem for both.

Theorem: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***not*** one-to-one if $n > m$.

Onto Linear Transformations

Definition: We say that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** or ***surjective*** if:

$$range(T) = \mathbb{R}^m.$$

We also say that T is a ***surjection*** or a ***covering*** (because T hits all the vectors of $\mathbb{R}^m$).

Theorem: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** ***if and only if*** $rank(T) = m$.

Example: Suppose $T_1, T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ are given by the following matrices with the corresponding rrefs. Describe the kernel and range of each, decide if either is one-to-one, and/or onto, and verify the Dimension Theorem for both.

$$[T_1] = \begin{bmatrix} -2 & -8 & 6 \\ 1 & 4 & -3 \end{bmatrix}, \quad \text{with rref} \quad R_1 = \begin{bmatrix} 1 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$[T_2] = \begin{bmatrix} -2 & -8 & 7 \\ 1 & 4 & -3 \end{bmatrix}, \quad \text{with rref} \quad R_2 = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***not*** onto if $n < m$.

Using the RREF of the Matrix of T

Theorem: Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and R is the rref of $[T]$.
Then:

1. T is ***one-to-one*** ***if and only if*** R does ***not*** have any free variables.

2. T is ***onto*** ***if and only if*** R does ***not*** have any row consisting only of zeroes.

Theorem — Equivalent Properties for Full-Rank Linear Transformations:

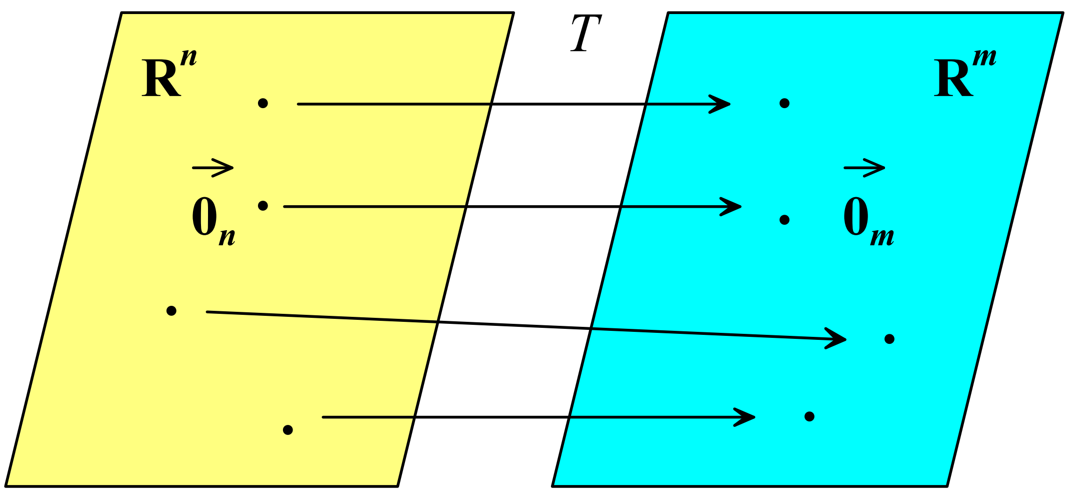
Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then:

1. if $m < n$: T is full-rank ***if only only if*** T is ***onto***.
2. if $m = n$: T is full-rank ***if and only if*** T is ***both one-to-one and onto***.
3. if $m > n$: T is full-rank ***if and only if*** T is ***one-to-one***.

Proof: Exercise.

A Recap of The One-to-One and Onto Properties

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if and only if $ker(T) = \left\{\vec{0}_n\right\}$. This means that if $\vec{v} \in \mathbb{R}^n$ is any other vector but $\vec{0}_n$, then $T(\vec{v}) \neq \vec{0}_m$.

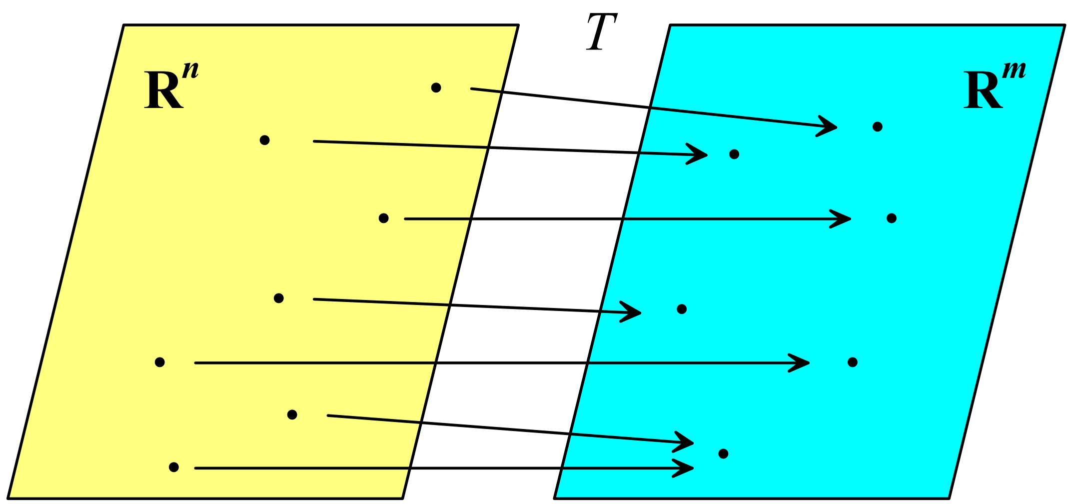


T is ***one-to-one*** if and only if:

$$ker(T) = \left\{\vec{0}_n\right\}$$

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** if and only if $range(T) = \mathbb{R}^m$.

This means that for any vector $\vec{w} \in \mathbb{R}^m$, we can find at least one vector $\vec{v} \in \mathbb{R}^n$ such that $T(\vec{v}) = \vec{w}$. We remark that more than one such vector $\vec{v}$ could exist for every $\vec{w}$. This also means that $rank(T) = m$.



T is ***onto*** if and only if:

$$range(T) = \mathbb{R}^m$$

Anything Can Happen:

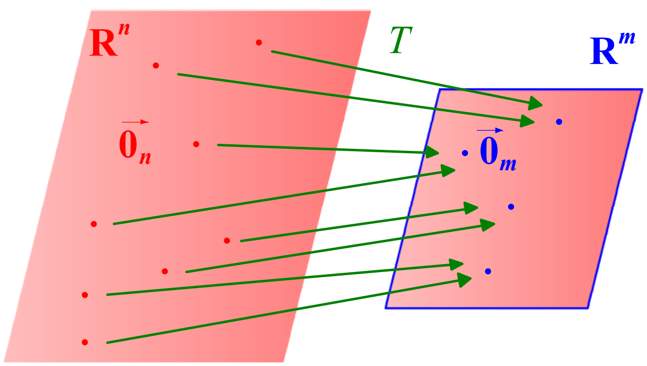
Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$.

If we ***don't*** know anything about n or m, then T can be:

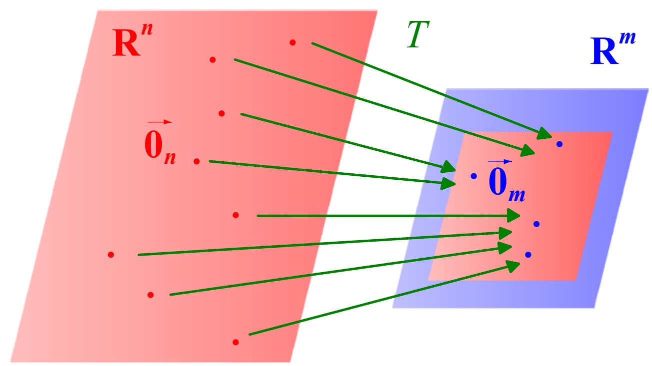
- one-to-one but not onto;
- onto but not one-to-one;
- ***neither*** one-to-one nor onto;
- ***both*** one-to-one and onto.

However, if we ***knew*** that:

- $n > m$, then T is automatically ***not one-to-one***; however, T can be onto, or not onto.



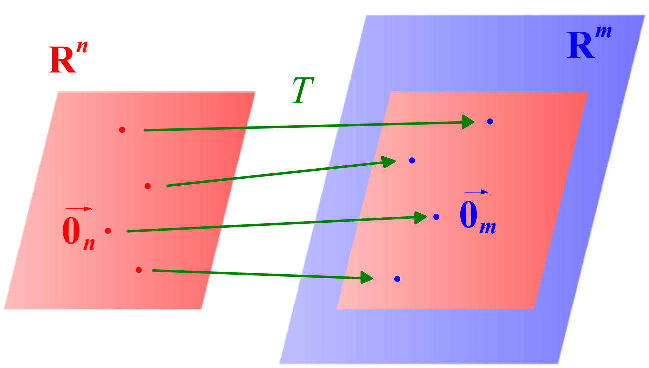
T is onto, but not one-to-one
$rank\,(T) = m$; T is full rank



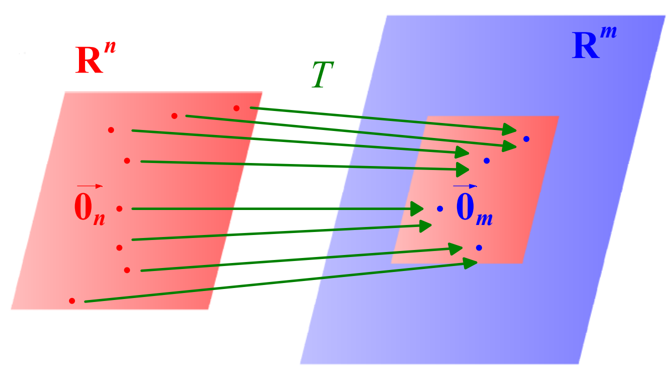
n > m

T is neither onto nor one-to-one
$rank\,(T) < m$

- $n < m$, then T is automatically ***not onto***; however, T can be one-to-one, or not one-to-one.



T is one-to-one, but not onto
$rank\,(T) = n$; T is full rank



n < m

T is neither one-to-one nor onto
$rank\,(T) < n$