3.2 Linearity Properties for Finite Sets of Vectors

Linear Combinations and Spans of Finite Sets of Vectors

Definition: Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors from a vector space $(V, \oplus, \odot)$, and

let $r_1,\ r_2, \ldots,\ r_n \in \mathbb{R}$. Then, a ***linear combination*** of the vectors $\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n$ with

coefficients $r_1,\ r_2,\ \ldots,\ r_n$ is an expression of the form:

$$(r_1 \odot \vec{v}_1) \oplus (r_2 \odot \vec{v}_2) \oplus \cdots \oplus (r_n \odot \vec{v}_n).$$

Similarly, the ***Span*** of the set of vectors $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is the set of ***all possible linear combinations*** of these vectors:

$$\begin{aligned} Span(S) &= Span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}) \\ &= \{(r_1 \odot \vec{v}_1) \oplus (r_2 \odot \vec{v}_2) \oplus \cdots \oplus (r_n \odot \vec{v}_n) \,| \\ &\qquad r_1, r_2, \ldots, r_n \in \mathbb{R}.\} \end{aligned}$$

Example: The vector space $\mathbb{P}^n$ consists of all polynomials of degree at most n.

Membership in A Span

A useful theorem:

Theorem — The Fundamental Theorem of Algebra:

Every non-constant polynomial $p(x)$ (that is, of degree $n \geq 1$), with complex (or possibly real) coefficients, has exactly n complex roots, counting multiplicities.

Consequence:

Theorem — Equality of Polynomials: Suppose that:

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \text{ and}$$

$$q(x) = d_0 + d_1 x + d_2 x^2 + \cdots + d_n x^n.$$

Then, ***as functions***, $p(x) = q(x)$ (i.e. the graphs of the two functions are the same) ***if and only if*** $c_0 = d_0$, $c_1 = d_1$, …, $c_n = d_n$.

Note: We say that $p(x) = q(x)$ ***as functions*** if the values of the two functions agree for all real numbers $a \in \mathbb{R}$, that is:

$$p(a) = q(a) \text{ for all } a \in \mathbb{R}.$$

In other words, they have the same ***graph***.

Example: Consider the set S of polynomials from $\mathbb{P}^3$:

$$S = \left\{ \begin{array}{c} 4x^3 - 7x^2 - 5x + 6,\ 2x^3 - 3x^2 - 7x + 3, \\ 10x^3 - 19x^2 + x + 15 \end{array} \right\}$$

Let $p(x) = 2x^3 - 6x^2 + 20x + 3$.

Decide whether or not $p(x)$ is a member of $Span(S)$.

Linear Independence of a Finite Set of Vectors

Definition: Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors from a vector space $(V, \oplus, \odot)$. We say that S is ***linearly independent*** if and only if the only solution to the equation:

$$(c_1 \odot \vec{v}_1) \oplus (c_2 \odot \vec{v}_2) \oplus \cdots \oplus (c_n \odot \vec{v}_n) = \vec{0}_V$$

is the trivial solution $c_1 = 0, c_2 = 0, \ldots, c_n = 0$. As before, we will refer to this equation as a ***dependence test equation*** and sometimes just say "independent" to mean linearly independent. The opposite of being linearly independent is being linearly dependent, which means there is a non-trivial solution to the dependence test equation, that is, where at least one c_i is non-zero.

Theorem: Let $(V, \oplus, \odot)$ be a vector space, and $\vec{v} \in V$. Then $S = \{\vec{v}\}$ is linearly independent ***if and only if*** $\vec{v} \neq \vec{0}_V$.

Theorem: Let $(V, \oplus, \odot)$ be a vector space, and $\vec{v}_1, \vec{v}_2 \in V$. Then $S = \{\vec{v}_1, \vec{v}_2\}$ is linearly independent ***if and only if*** $\vec{v}_1$ is not parallel to $\vec{v}_2$.

Theorem: Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors from a vector space $(V, \oplus, \odot)$. Then: S is linearly ***dependent if and only if*** at least one vector (which, without loss of generality, we can set to be $\vec{v}_1$) is a linear combination of $\vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n$, that is:

$$\vec{v}_1 = (r_2 \odot \vec{v}_2) \oplus (r_3 \odot \vec{v}_3) \oplus \cdots \oplus (r_n \odot \vec{v}_n),$$

for some scalars $r_2, r_3, \ldots, r_n \in \mathbb{R}$.

A Sufficient Test for Independence of Sets of Polynomials

Theorem: Suppose $S = \{p_1(x), p_2(x), \ldots, p_k(x)\}$ is a set of polynomials from $\mathbb{P}^n$ with ***distinct*** degrees. Then S is linearly independent. In particular, the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent.

Example: Consider the set $S = \{e^{-2x}, e^x, e^{5x}\}$.

Generalization:

Theorem: Suppose $S = \{e^{k_1 x}, e^{k_2 x}, \ldots, e^{k_n x}\}$,
where $k_1 < k_2 < \cdots < k_n$ are n ***distinct*** real numbers.
Then: S is linearly independent.