3.6 Coordinate Vectors and Matrices for Linear Transformations

Definition: Let $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ be an ***ordered basis*** for a finite dimensional vector space V. If $\vec{v}$ is any vector in V, we know that $\vec{v}$ can be expressed ***uniquely*** as a linear combination of the vectors of B :

$$\vec{v} = c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_n\vec{w}_n.$$

We call the vector $\langle c_1, c_2, \ldots, c_n \rangle$
the ***coordinate vector of $\vec{v}$ with respect to*** B, written as:

$$\langle \vec{v} \rangle_B = \langle c_1, c_2, \ldots, c_n \rangle.$$

The $n \times 1$ matrix corresponding to $\langle \vec{v} \rangle_B$ is called
the ***coordinate matrix of $\vec{v}$ with respect to*** B, written as:

$$[\vec{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Example:

Let $B = \{\langle -1, 0, 1\rangle, \langle 1, 1, -1\rangle, \langle 0, -1, -1\rangle\}$,
$\vec{v} = \langle 7, -3, -2\rangle$.

Find $[\vec{v}]_B$.

Theorem: For any ordered basis $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ of an n −dimensional vector space V, the function $T : V \to \mathbb{R}^n$ given by:

$$T(\vec{v}) = \langle \vec{v} \rangle_B$$

is a ***linear transformation***. In particular, if $V = \mathbb{R}^n$ and B is a basis for $\mathbb{R}^n$, then T is in fact one-to-one and onto, i.e., an ***isomorphism of*** $\mathbb{R}^n$.

Proof: Suppose that

$\langle \vec{u} \rangle_B = \langle c_1, c_2, \ldots, c_n \rangle$, and

$\langle \vec{v} \rangle_B = \langle d_1, d_2, \ldots, d_n \rangle$.

These mean that:

Coordinates for $\mathbb{P}^n$

Let $p(x) = 5x^2 - 3x + 7$. Find $[p(x)]_B$, where:

a) $B = \{1, x, x^2\}$.

b) $B = \{x^2 - 5, x + 2, x - 1\}$.

Coordinate Vectors for $W = Span(B)$

Example: Consider $B = \{\sin(x), \cos(x)\}$ and $W = Span(B)$. Find $[f(x)]_B$, for the following functions:

a) $f(x) = 5\sin(x) - 8\cos(x)$

b) $f(x) = \sin\left(x + \frac{\pi}{4}\right)$

c) $f(x) = \cos(x + \sin^{-1}(3/5))$

Constructing A Matrix For T

Definition/Theorem: Let $T : V \to W$ be a linear transformation, where $dim(V) = n$ and $dim(W) = m$. Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for V, and let $B' = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ be a basis for W. The $m \times n$ matrix $[T]_{B,B'}$, given by:

$$[T]_{B,B'} = \Big[[T(\vec{v}_1)]_{B'} \mid [T(\vec{v}_2)]_{B'} \mid \cdots \mid [T(\vec{v}_n)]_{B'} \Big],$$

is called ***the matrix of*** T ***relative to*** B ***and*** B'.

For any $\vec{v} \in V$, we can compute $T(\vec{v})$ via:

$$[T(\vec{v})]_{B'} = [T]_{B,B'}[\vec{v}]_B.$$

If $T : V \to V$ is an ***operator*** and we use the same basis B for the domain and codomain (that is, $B = B'$), we simply write $[T]_B$ instead of $[T]_{B,B}$.

How to Use the Matrix for *T*

ENCODE :

Given $\vec{v} \in V$, find $[\vec{v}]_B \in \mathbb{R}^n$.

MULTIPLY :

Compute the product $[T]_{B,B'}[\vec{v}]_B = [T(\vec{v})]_{B'} \in \mathbb{R}^m$.

DECODE :

Use the coefficients of $[T(\vec{v})]_{B'}$ and the basis B' to explicitly find $T(\vec{v}) \in W$.

Example: Let $T : \mathbb{P}^3 \to \mathbb{P}^2$ be the operator given by:

$$T(p(x)) = 3p'(x) + 7xp''(x) + p(-1) \cdot x^2.$$

Warm-up: Compute $T(2 + 8x - 5x^2 + 4x^3)$

Explain why $T(p(x)) \in \mathbb{P}^2$ for any $p(x) \in \mathbb{P}^3$.

Prove that T is indeed a linear transformation.

Let $B = \{1, x, x^2, x^3\}$ be the standard basis for $\mathbb{P}^3$, and $B' = \{1, x, x^2\}$ the standard basis for $\mathbb{P}^2$.

Find $[T]_{B,B'}$.

Recompute $T(2 + 8x - 5x^2 + 4x^3)$ using $[T]_{B,B'}$.

Example: Let us suppose that we are given a linear transformation $T : \mathbb{P}^2 \to \mathbb{P}^1$, with matrix:

$$[T]_{B,B'} = \begin{bmatrix} 2 & -3 & 5 \\ 4 & 1 & -2 \end{bmatrix},$$

where $B = \{x^2 + 5, x - 2, 1\}$ and $B' = \{x + 1, x - 1\}$.

Find $T(7x^2 + 4x - 8)$.

Function Spaces Preserved by the Derivative

Example: Find the matrix of the derivative operator D applied to the function space:

$$V = Span(\{x^2e^{4x}, xe^{4x}, e^{4x}\})$$

Revisiting Projections

Example: Suppose that Π is the plane with equation: $5x + 2y - 6z = 0$.

Find $[proj_{\Pi}]$.