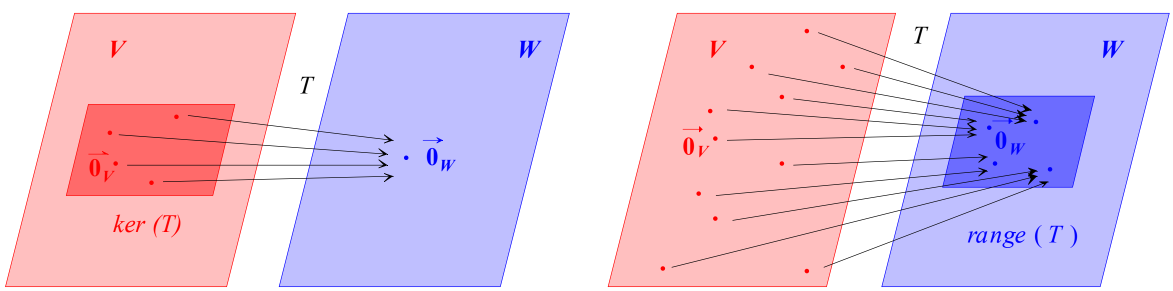
3.7 One-to-One and Onto Linear Transformations ; Compositions of Linear Transformations

Review:



$$ker(T) = \left\{ \vec{v} \in V \mid T(\vec{v}) = \vec{0}_W \right\}$$

$$range(T) = \left\{ \vec{w} \in W \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V \right\}$$

One-to-One Transformations

Definition: We say that a linear transformation $T : V \to W$ is ***one-to-one*** or ***injective*** if the image of different vectors from the domain are different vectors from the codomain:

$$\text{If } \vec{v}_1 \neq \vec{v}_2 \text{ then } T(\vec{v}_1) \neq T(\vec{v}_2).$$

We again say that T is an ***injection*** or an ***embedding***.

Theorem: A linear transformation $T : V \to W$ is ***one-to-one*** if and only if the only way two vectors from the domain have the same image in the codomain is for them to be the same vector to begin with:

$$\text{If } T(\vec{v}_1) = T(\vec{v}_2) \text{ then } \vec{v}_1 = \vec{v}_2.$$

In other words, the only solution to $T(\vec{v}_1) = T(\vec{v}_2)$ is $\vec{v}_1 = \vec{v}_2$.

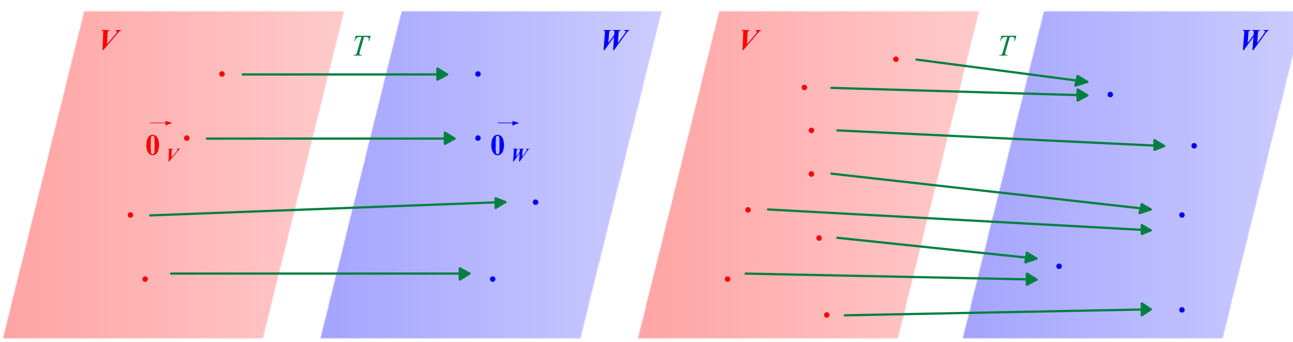
Theorem: A linear transformation $T : V \to W$ is one-to-one ***if and only if*** $ker(T) = \left\{\vec{0}_V\right\}$.

Onto Transformations

Definition: We say that a linear transformation $T : V \rightarrow W$ is ***onto*** or ***surjective*** if the range of T is ***all*** of W :

$$range(T) = W.$$

Since $rank(T) = dim(range(T))$, we can also say that T is ***onto*** ***if and only if*** $rank(T) = dim(W)$, in the case when W is ***finite dimensional***.



one-to-one $\Leftrightarrow$

$$ker(T) = \left\{\vec{0}_V\right\}$$

onto $\Leftrightarrow$

$$range(T) = W$$

Finding the Kernel and Range Using $[T]_{B,B'}$

The information provided by $[T]_{B,B'}$ and its rref simply needs to be ***decoded*** with respect to the appropriate basis: we use B for $ker(T)$ and B' for $range(T)$.

Theorem: Suppose that $T : V \to W$ is a linear transformation, with $dim(V) = n$ and $dim(W) = m$, both finite-dimensional vector spaces. Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for V, and let $B^{/} = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ be a basis for W. Let us construct the $m \times n$ matrix $[T]_{B,B^{/}}$ as we did in the previous Section, and let R be the rref of $[T]_{B,B^{/}}$. Suppose that:

$$\{\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_k\} \subset \mathbb{R}^n$$

is the basis that we obtain for $nullspace\left([T]_{B,B^{/}}\right)$ using R, as we did in Chapter 2.

By the Uniqueness of Representation Property, we know that there exists $\vec{u}_i \in V$ so that $\langle \vec{u}_i \rangle_B = \vec{z}_i$ for every $i = 1 \ldots k$.

We conclude that the set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\} \subset V$ is a ***basis*** for $ker(T)$.

As usual, if there are no free variables in R, then $nullspace\left([T]_{B,B^{/}}\right) = \left\{\vec{\mathbf{0}}_n\right\}$,

and consequently, $ker(T) = \left\{\vec{\mathbf{0}}_V\right\}$.

Similarly, the set of original columns:

$$\{\vec{c}_{i_1}, \vec{c}_{i_2}, \ldots, \vec{c}_{i_r}\} \subset \mathbb{R}^m$$

from $[T]_{B,B'}$ corresponding to the leading 1's of R form a basis for $columnspace\left([T]_{B,B'}\right)$ as we found in Chapter 2, and there exists $\vec{d}_j \in W$ so that $\left\langle \vec{d}_j \right\rangle_{B'} = \vec{c}_{i_j}$ for every $j = 1 \ldots r$.

We conclude that the set $\left\{\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_r\right\} \subset W$ is a ***basis*** for $range(T)$.

If T is the zero transformation, then $range(T) = \left\{\vec{\mathbf{0}}_W\right\}$.

The Dimension Theorem for Abstract Vector Spaces

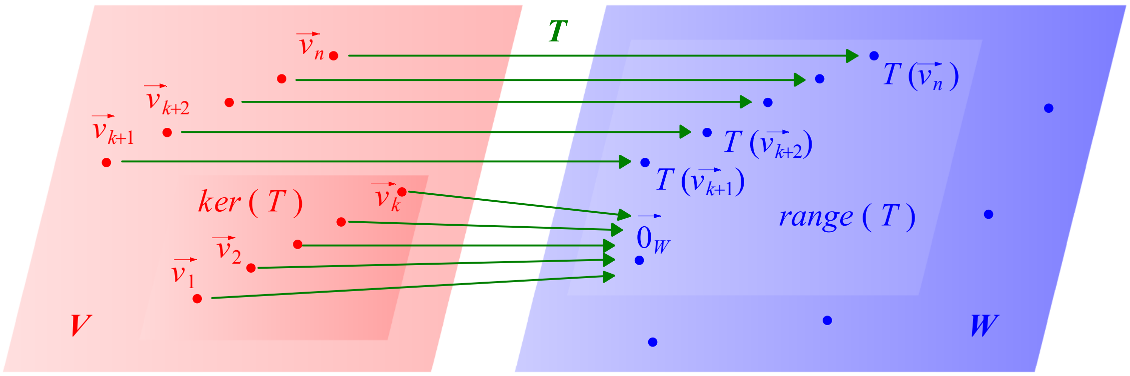
Theorem — The Dimension Theorem:

Let $T : V \to W$ be a linear transformation, and suppose that V ***is finite dimensional*** with $dim(V) = n$. Then, both $ker(T)$ and $range(T)$ are finite dimensional, and we can define:

$$rank(T) = dim(range(T)), \text{ and}$$
$$nullity(T) = dim(ker(T)).$$

Furthermore, these quantities are related by the equation:

$$rank(T) + nullity(T)$$
$$= n = dim(V) = dim\left(domain \text{ of } T\right).$$



Idea of the Proof

Let's say $nullity(T) = k$.

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ a basis for $ker(T)$

Use the ***Extension Theorem***:

$B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n\}$ a basis for V.

Examine $range(T)$:

$$\begin{aligned}\vec{v} = {} & c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k + \\ & c_{k+1}\vec{v}_{k+1} + c_{k+2}\vec{v}_{k+2} + \cdots + c_n\vec{v}_n.\end{aligned}$$

$$\begin{aligned}T(\vec{v}) = {} & c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_k T(\vec{v}_k) + \\ & c_{k+1} T(\vec{v}_{k+1}) + c_{k+2} T(\vec{v}_{k+2}) + \cdots + c_n T(\vec{v}_n) \\ = {} & c_{k+1} T(\vec{v}_{k+1}) + c_{k+2} T(\vec{v}_{k+2}) + \cdots + c_n T(\vec{v}_n)\end{aligned}$$

This tells us that every vector in $range(T)$ is . . . .

Show that $\{T(\vec{v}_{k+1}), T(\vec{v}_{k+2}), \ldots, T(\vec{v}_n)\}$ is linearly independent:

$$d_{k+1}T(\vec{v}_{k+1}) + d_{k+2}T(\vec{v}_{k+2}) + \cdots + d_nT(\vec{v}_n) = \vec{0}_W.$$

$$T(d_{k+1}\vec{v}_{k+1} + d_{k+2}\vec{v}_{k+2} + \cdots + d_n\vec{v}_n) = \vec{0}_W.$$

Conclusion: the vector inside the parenthesis is in . . .

$$
\begin{aligned}
& d_{k+1}\vec{v}_{k+1} + d_{k+2}\vec{v}_{k+2} + \cdots + d_n\vec{v}_n \\
= \ & d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_k\vec{v}_k,
\end{aligned}
$$

$$
\begin{gathered}
-d_1\vec{v}_1 - d_2\vec{v}_2 - \cdots - d_k\vec{v}_k + \\
d_{k+1}\vec{v}_{k+1} + d_{k+2}\vec{v}_{k+2} + \cdots + d_n\vec{v}_n = \vec{0}_W
\end{gathered}
$$

Conclusion:

Comparing Dimensions

Theorem: Suppose $T : V \to W$ is a linear transformation of finite dimensional vector spaces. Then:

a) if $dim(V) < dim(W)$, then T ***cannot*** be onto.

b) if $dim(V) > dim(W)$, then T ***cannot*** be one-to-one.

Compositions of Linear Transformations

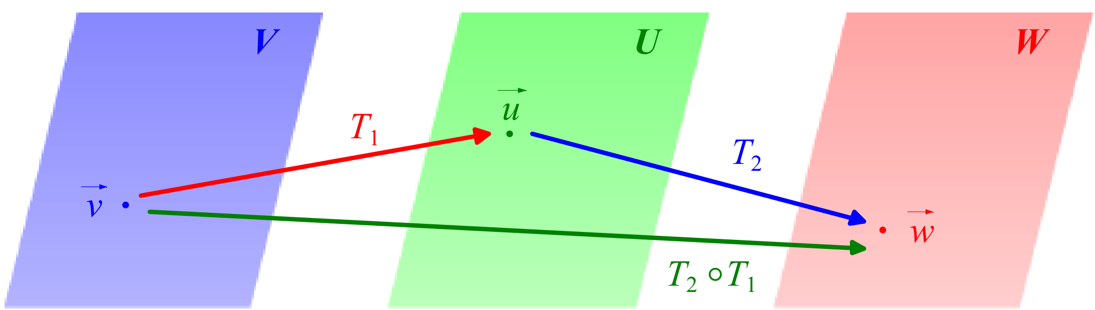
Definition/Theorem: Suppose $T_1 : V \to U$, and $T_2 : U \to W$ are linear transformations. The ***composition:***

$$T_2 \circ T_1 : V \to W$$

is again a linear transformation, with action given as follows:

Suppose $\vec{v} \in V$, $T_1(\vec{v}) = \vec{u} \in U$, and $T_2(\vec{u}) = \vec{w} \in W$. Then:

$$(T_2 \circ T_1)(\vec{v}) = T_2(T_1(\vec{v})) = T_2(\vec{u}) = \vec{w}.$$



The Composition of

Two Linear Transformations

The Matrix of a Composition

Theorem: Let $T_1 : V \to U$ and $T_2 : U \to W$ be linear transformations of finite dimensional vector spaces. Let B be a basis for V, $B^/$ a basis for U, and $B^{//}$ a basis for W. Then:

$$[T_2 \circ T_1]_{B,B^{//}} = [T_2]_{B^/,B^{//}} \cdot [T_1]_{B,B^/}.$$

Proof:

$$B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$

$$B^/ = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$$

$$B^{//} = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$$

$$
\begin{aligned}
&[T_2 \circ T_1]_{B,B''} \\
&= Z \\
&= \Big[\ [(T_2 \circ T_1)(\vec{v}_1)]_{B''} \ \mid \ [(T_2 \circ T_1)(\vec{v}_2)]_{B''} \ \Big| \\
&\qquad \cdots \ [(T_2 \circ T_1)(\vec{v}_n)]_{B''} \ \Big]
\end{aligned}
$$

$$
\begin{aligned}
&[T_2]_{B',B''} \\
&= X \\
&= \Big[\ [T_2(\vec{u}_1)]_{B''} \ \mid \ [T_2(\vec{u}_2)]_{B''} \ \mid \cdots \Big| \ \ [T_2(\vec{u}_k)]_{B''} \Big]
\end{aligned}
$$

$$
\begin{aligned}
&[T_1]_{B,B'} \\
&= Y \\
&= \Big[\ [T_1(\vec{v}_1)]_{B'} \ \mid \ [T_1(\vec{v}_2)]_{B'} \ \mid \cdots \Big| \ \ [T_1(\vec{v}_n)]_{B'} \Big]
\end{aligned}
$$

Function Spaces Preserved by the Derivative

Example: Find the matrix of the ***second derivative***, D^2, applied to the function space:

$$V = Span(\{x^2e^{4x}, xe^{4x}, e^{4x}\})$$