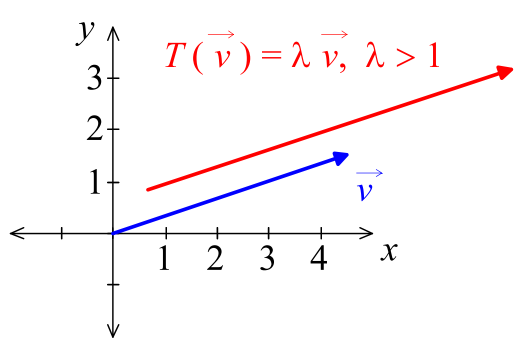
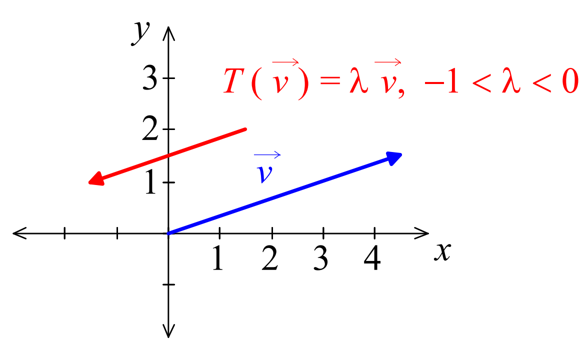
6.1 Eigentheory of Matrices

Definition: Let A be an $n \times n$ matrix. We say that $\lambda \in \mathbb{R}$ (the Greek letter ***lambda***) is an ***eigenvalue*** of A, and a ***non-zero*** vector $\vec{v} \in \mathbb{R}^n$ is an ***eigenvector*** for A associated to λ, or simply an eigenvector for λ, if:

$$A\vec{v} = \lambda\vec{v}.$$

In other words, if $T : \mathbb{R}^n \to \mathbb{R}^n$, with $[T] = A$,
then $T(\vec{v})$ is ***parallel*** to $\vec{v}$:





How do we Find Eigenvalues and Eigenvectors?

$$A\vec{v} = \lambda\vec{v} = (\lambda I_n)(\vec{v})$$

$$(\lambda I_n)(\vec{v}) - A\vec{v} = \vec{0}_n$$

$$(\lambda I_n - A)\vec{v} = \vec{0}_n$$

Definition/Theorem: Let A be an $n \times n$ matrix. Then we can find a real number λ and a non-zero vector $\vec{v} \in \mathbb{R}^n$ such that:

$$A\vec{v} = \lambda\vec{v}$$

if and only if $det(\lambda I_n - A) = 0$.

The equation above is called the *characteristic equation* of the matrix A.

The determinant in this equation is a polynomial whose highest term is λ^n, and it is called the *characteristic polynomial* of A, denoted $p_A(\lambda)$, or $p(\lambda)$:

$$p_A(\lambda) = p(\lambda) = det(\lambda I_n - A).$$

The ***highest term*** comes from:

$$sgn(\sigma)(\lambda - a_{1,1}) \cdot (\lambda - a_{2,2}) \cdot \cdots \cdot (\lambda - a_{n,n})$$

But for this term, $\sigma = (1, 2, \ldots, n)$

$sgn(\sigma) = 1.$

Possibilities for 2×2 *Matrices*

Suppose A has integer or rational entries only.

The quadratic characteristic polynomial will have integer or rational coefficients.

It can have:

- ***two distinct roots***, (if the discriminant is not 0) or
- a ***single*** (repeated or double) root (if the discriminant is 0).

If the roots are ***distinct***, then they could be:

- ***imaginary***, (if the discriminant is negative)
- ***irrational***, (if the discriminant is a positive non-square)
- ***integer*** or ***rational***, (if the discriminant is a positive perfect square)

Example:

$$\begin{bmatrix} 285 & 504 \\ -160 & -283 \end{bmatrix}$$

Eigenspaces

Definition/Theorem — Eigenspaces:

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{R}$. We define the ***eigenspace*** of A associated to λ, denoted $Eig(A, \lambda)$, to be:

$$Eig(A, \lambda) = \{\vec{v} \in \mathbb{R}^n \,|\, A\vec{v} = \lambda\vec{v}\}.$$

Notice that $A\vec{0}_n = \vec{0}_n = \lambda\vec{0}_n$, so $\vec{0}_n \in Eig(A, \lambda)$.

If λ is an actual ***eigenvalue*** for A, then $Eig(A, \lambda) = nullspace(A - \lambda I_n)$, which is a ***non-zero subspace*** of $\mathbb{R}^n$ containing all the ***eigenvectors*** of A associated to λ, and thus its dimension is strictly ***positive***.

If λ is ***not*** an eigenvalue of A, then $Eig(A, \lambda)$ consists only of $\vec{0}_n$. In this case, we can refer to $Eig(A, \lambda)$ as a ***trivial eigenspace***.

Thus, we can say that λ is an ***eigenvalue*** of A ***if and only if*** the eigenspace $Eig(A, \lambda)$ is at least ***1-dimensional***.

Eigentheory for Triangular Matrices

Theorem: Let A be an upper or lower ***triangular*** $n \times n$ matrix, and suppose the entries along the main diagonal are $c_1, c_2, \ldots, c_n$. Then: the ***characteristic polynomial*** of A is:

$$p(\lambda) = (\lambda - c_1)(\lambda - c_2)\cdots(\lambda - c_n),$$

and therefore the ***eigenvalues*** are precisely $c_1, c_2, \ldots, c_n$. Moreover, if:

$$D = Diag(d_1, d_2, \ldots, d_n)$$

is a ***diagonal*** matrix, then for every $i = 1 \ldots n$: $\vec{e}_i$ is an ***eigenvector*** for d_i.

Example:

$$\begin{bmatrix} 5 & 13 & -6 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

Follow up: what happens if the 14 is turned into 13?