7.2 Geometric Constructions in Inner Product Spaces

Further Properties of Inner Products

Theorem: Let V be an inner product space under the bilinear form $\langle \mid \rangle$. Then the following properties also hold, for all vectors $\vec{u}$, $\vec{v}$ and $\vec{w} \in V$:

1. $\langle \vec{u} \mid k \cdot \vec{v} \rangle = k \cdot \langle \vec{u} \mid \vec{v} \rangle$

2. $\langle \vec{u} \mid \vec{v} + \vec{w} \rangle = \langle \vec{u} \mid \vec{v} \rangle + \langle \vec{u} \mid \vec{w} \rangle$

3. $\langle \vec{u} - \vec{v} \mid , \vec{w} \rangle = \langle \vec{u} \mid \vec{w} \rangle - \langle \vec{v} \mid \vec{w} \rangle$

4. $\langle \vec{u} \mid \vec{v} - \vec{w} \rangle = \langle \vec{u} \mid \vec{v} \rangle - \langle \vec{u} \mid \vec{w} \rangle$

5. $\langle \vec{u} + \vec{v} \mid \vec{u} + \vec{v} \rangle = \langle \vec{u} \mid \vec{u} \rangle + 2\langle \vec{u} \mid \vec{v} \rangle + \langle \vec{v} \mid \vec{v} \rangle$

6. $\langle \vec{u} - \vec{v} \mid \vec{u} - \vec{v} \rangle = \langle \vec{u} \mid \vec{u} \rangle - 2\langle \vec{u} \mid \vec{v} \rangle + \langle \vec{v} \mid \vec{v} \rangle$

7. $\langle \vec{u} + \vec{v} \mid \vec{u} - \vec{v} \rangle = \langle \vec{u} \mid \vec{u} \rangle - \langle \vec{v} \mid \vec{v} \rangle$

8. $\langle \vec{u} \mid \vec{\mathbf{0}}_V \rangle = 0 = \langle \vec{\mathbf{0}}_V \mid \vec{u} \rangle$

Norms and Distances

Definition: Let $\vec{v}$, $\vec{u} \in V$. Define the ***norm*** or the ***length*** of $\vec{v}$ by:

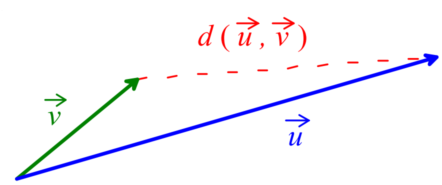
$$\|\vec{v}\| = \sqrt{\langle \vec{v} \mid \vec{v} \rangle}, \text{ in other words:}$$

$$\|\vec{v}\|^2 = \langle \vec{v} \mid \vec{v} \rangle$$

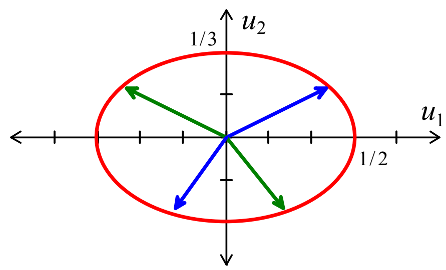
In particular, we say that $\vec{v}$ is a ***unit vector*** if $\|\vec{v}\| = 1$. The set of all unit vectors in V is called the ***unit sphere*** or ***unit circle*** of V.

We can also define the ***distance*** between two vectors by:

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$



Unit Vectors



Some Vectors on The "Unit Circle" of $\mathbb{R}^2$

under $\langle \vec{u} \mid \vec{v} \rangle = 4u_1v_1 + 9u_2v_2$

Properties of Norms and Distances

Theorem: For all vectors $\vec{u}$, $\vec{v}$ in an inner product space V, and all $k \in \mathbb{R}$:

1. $\|k \cdot \vec{u}\| = |k| \cdot \|\vec{u}\|$

2. $d(\vec{u},\vec{v}) = d(\vec{v},\vec{u})$

3. $d(k \cdot \vec{u}, k \cdot \vec{v}) = |k| \cdot d(\vec{u},\vec{v})$

The Cauchy-Schwarz Inequality

Theorem (The Cauchy-Schwarz Inequality):

For all vectors $\vec{u}$ and $\vec{v}$ from an inner product space V :

$$\left| \left\langle \vec{u} \mid \vec{v} \right\rangle \right| \leq \|\vec{u}\| \cdot \|\vec{v}\|$$

or equivalently:

$$\left\langle \vec{u} \mid \vec{v} \right\rangle^2 \leq \left\langle \vec{u} \mid \vec{u} \right\rangle \cdot \left\langle \vec{v} \mid \vec{v} \right\rangle$$

Angle Between Vectors

$$\frac{\left|\langle \vec{u} \mid \vec{v} \rangle\right|}{\|\vec{u}\| \, \|\vec{v}\|} \leq 1$$

$$-1 \leq \frac{\langle \vec{u} \mid \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|} \leq 1.$$

Define for two non-zero vectors:

$$\cos(\theta) = \frac{\langle \vec{u} \mid \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|}$$

In particular, if $\cos(\theta) = 0 = \langle \vec{u} \mid \vec{v} \rangle$, then $\theta = \pi/2$, and we will say that $\vec{u}$ and $\vec{v}$ are ***orthogonal*** to each other.

We will agree that $\vec{\mathbf{0}}_V$ is orthogonal to any other vector.

Definitions: If $\vec{u}$ and $\vec{v}$ are non-zero vectors in V, we define the ***angle*** between them as the angle θ, where $0 \leq \theta \leq \pi$, such that:

$$\cos(\theta) = \frac{\left\langle \vec{u} \mid \vec{v} \right\rangle}{\|\vec{u}\| \, \|\vec{v}\|}$$

Furthermore, we will say that $\vec{u}$ is ***orthogonal*** to $\vec{v}$ if and only if $\left\langle \vec{u} \mid \vec{v} \right\rangle = 0$. We write this symbolically as:

$$\vec{u} \perp \vec{v} \iff \left\langle \vec{u} \mid \vec{v} \right\rangle = 0.$$

In particular, $\vec{\mathbf{0}}_V$ is orthogonal to ***all*** vectors in V.

The Triangle Inequalities

Theorem (The Triangle Inequality — Norm Version):

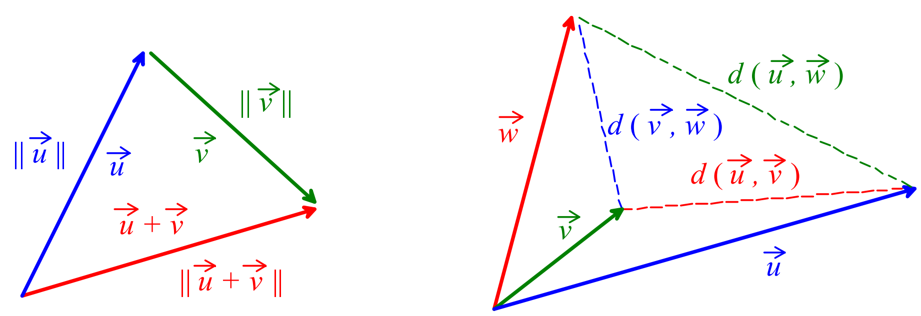
For any two vectors $\vec{u}$ and $\vec{v}$ in an inner product space V :

$$\|\vec{u} + \vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$$

Theorem: (The Triangle Inequality — Distance Version):

For any three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in an inner product space V :

$$d(\vec{u},\vec{v}) \le d(\vec{u},\vec{w}) + d(\vec{w},\vec{v})$$



Flashback from Ancient Greece

Theorem (The Generalized Pythagorean Theorem):
If $\vec{u}$ and $\vec{v}$ are ***orthogonal*** vectors in an inner product space V, then:

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u} + \vec{v}\|^2.$$