A Portrait of Linear Algebra

Selected Answers to the Exercises

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Chapter Zero Exercises

1. A True logical statement.
2. A logical statement, but it is False, because $-5 < 3$ but $25 > 9$.
3. A True logical statement, using the properties of inequalities found in Appendix A.
4. A False logical statement, because if $x < 0$, then $\sqrt{x}$ is imaginary.
5. A True logical statement as of June 2009, with 237 consecutive weeks.
6. Not a logical statement, because it cannot be ascertained to be True or False ("best" is not a well-defined adjective; unlike the previous Exercise, where "most number of consecutive weeks as number 1" is well defined).
7. Converse: If you can watch TV tonight, then you did your homework before dinner. Inverse: If you do not do your homework before dinner, you cannot watch TV tonight. Contrapositive: If you cannot watch TV tonight, then you did not do your homework before dinner.
8. Converse: If we don't go to the beach tomorrow, then it rained. Inverse: If it doesn't rain tomorrow, we will go to the beach. Contrapositive: If we go to the beach tomorrow, then it did not rain.
9. Converse: If $\cos(x) \geq 0$, then $0 \leq x \leq \pi/2$. Inverse: If $x > \pi/2$ or $x < 0$, then $\cos(x) < 0$. Contrapositive: If $\cos(x) < 0$, then $x > \pi/2$ or $x < 0$.
10. If $f(x)$ is continuous on the closed interval $[a,b]$ then it possesses both a maximum and a minimum on $[a,b]$. Converse: If $f(x)$ possesses both a maximum and a minimum on $[a,b]$, then $f(x)$ is continuous on $[a,b]$. Inverse: If $f(x)$ is not continuous on $[a,b]$, then $f(x)$ either does not possess an absolute maximum or an absolute minimum on $[a,b]$. Contrapositive: If $f(x)$ does not possess either an absolute maximum or an absolute minimum on $[a,b]$, then $f(x)$ is not continuous at $x = a$.
11. $A \cup B = \{a,b,c,f,g,h,i,j,m,p,q\}$, $A \cap B = \{c,h,j\}$, $A - B = \{a,f,i,m\}$, $B - A = \{b,g,p,q\}$.
12. $A \cup B = \{a,b,d,g,h,j,k,p,q,r,s,t,v\}$, $A \cap B = \{d,g,h,p,t\}$, $A - B = \{a,j,r\}$, $B - A = \{b,k,q,s,v\}$.
23. If there were a largest positive number x, what can you say about $x + 1$?
27. "If n does not have a prime factor which is at most $\sqrt{n}$, then n is prime." The number 11303 is composite. One prime factor is smaller than 100.
38. 2027 and 2029. 39. 233 49. Hint: In Step 3, write 2^{n+1} as $2(2^n) = 2^n + 2^n$.
54. f. For any two sets X and Y : $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$. 55. a. $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$
58. a. $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$; 8 subsets.
    c. you get exactly the same list as the subsets on the right column.

Chapter One Exercises

1.1 Exercises

1. These are found in the Key Concepts.
2. b. $\|\vec{u}\| = \sqrt{65}$; c. $\vec{u}_1 = \dfrac{1}{\sqrt{65}}\langle -4,7\rangle$ and $\vec{u}_2 = \dfrac{-1}{\sqrt{65}}\langle -4,7\rangle$ d. $3\vec{v} = \langle 9,15\rangle$, $5\vec{w} = \langle 5,-10\rangle$, $\vec{v}+5\vec{w} = \langle 14,5\rangle$ and $3\vec{v}-5\vec{w} = \langle 4,25\rangle$
3. b. $2\vec{u} = \langle 10,-6,4\rangle$, $3\vec{w} = \langle -6,15,12\rangle$, $2\vec{u}+3\vec{w} = \langle 4,9,16\rangle$ and $2\vec{u}-3\vec{w} = \langle 16,-21,-8\rangle$
   c. $\|\vec{w}\| = \sqrt{45} = 3\sqrt{5}$ d. $\vec{u}_1 = \dfrac{1}{3\sqrt{5}}\langle -2,5,4\rangle$ and $\vec{u}_1 = \dfrac{-1}{3\sqrt{5}}\langle -2,5,4\rangle$.
   e. i. $-\dfrac{3}{5}\vec{w} = \langle 6/5,-3,-12/5\rangle$ ii. $2\vec{u}+5\vec{v} = \langle 30,-6,-31\rangle$
   iii. $3\vec{w}-4\vec{u} = \langle -26,27,4\rangle$ iv. $-4\vec{u}+7\vec{v}-2\vec{w} = \langle 12,2,-65\rangle$.
4. a. $\vec{u}+\vec{v} = \langle 1,-2,7,3\rangle$ b. $\vec{u}+\vec{w} = \langle -1,-3,4,-2\rangle$ c. $\vec{v}-\vec{w} = \langle 2,1,3,5\rangle$
   d. $-2\vec{u} = \langle -6,10,-2,-14\rangle$ e. $\dfrac{3}{4}\vec{v} = \left\langle -\dfrac{3}{2},\dfrac{9}{4},\dfrac{9}{2},-3\right\rangle$ f. $-\dfrac{5}{3}\vec{w} = \left\langle \dfrac{20}{3},-\dfrac{10}{3},-5,15\right\rangle$
   g. $5\vec{u}+3\vec{v} = \langle 9,-16,23,23\rangle$ h. $-\dfrac{3}{2}\vec{u}+\dfrac{5}{4}\vec{v} = \left\langle -7,\dfrac{45}{4},6,-\dfrac{31}{2}\right\rangle$
   i. $2\vec{u}-3\vec{v}+7\vec{w} = \langle -16,-5,5,-37\rangle$ j. $-5\vec{u}+2\vec{v}-4\vec{w} = \langle -3,23,-5,-7\rangle$
   k. $-\dfrac{3}{2}\vec{u}+\dfrac{3}{4}\vec{v}-\dfrac{5}{3}\vec{w} = \left\langle \dfrac{2}{3},\dfrac{77}{12},-2,\dfrac{3}{2}\right\rangle$ l. $\dfrac{3}{2}\vec{u}-\dfrac{3}{4}\vec{v}+2\vec{w} = \left\langle -2,-\dfrac{23}{4},3,-\dfrac{9}{2}\right\rangle$
5. $\vec{u} = \langle -15,6,7\rangle$ and $\vec{v} = \langle 42,-17,-16\rangle$. 6. Yes: $\langle -3,7\rangle = 40\langle 5,-2\rangle + 29\langle -7,3\rangle$.
7. Yes: $\langle -17,-9,29,-37\rangle = 5\langle 3,-5,1,7\rangle + 8\langle -4,2,3,-9\rangle$.
8. No: Using the first two coordinates, we get $x = -4$ and $y = 9$, but although these satisfy the 3rd coordinate, they do not satisfy the 4th.
9. $\vec{u} = \langle -3,4,2,6,-7\rangle$ and $\vec{v} = \langle -1,-3,5,-3,2\rangle$. 10. $(7,-3)$
11. $(-4,1,7)$ 12. $\vec{u} = \langle -4,4,-8\rangle$
22. Contrapositive: if $\vec{u} = \langle u_1,u_2\rangle$ and $\vec{v} = \langle v_1,v_2\rangle$ are vectors in $\mathbb{R}^2$, then they are ***not parallel*** to each other ***if and only if*** $u_1v_2 - u_2v_1 \neq 0$.
35. PQ is 26 cm. long.

1.2 Exercises

1. $y = 4x/7$ 2. $y = -5x/3$ 3. $x = 5t,\ y = -4t,\ z = 2t$, and $t = x/5 = y/(-4) = z/2$.
4. $x = -t,\ y = 3t,\ z = -6t$, and $t = -x = y/3 = z/(-6)$. 5. $7x+5y = 6$
6. $x = 2-3t,\ y = -7+6t,\ z = 4+8t$, and $t = \dfrac{x-2}{-3} = \dfrac{y+7}{6} = \dfrac{z-4}{8}$
7. $x = 3+2t,\ y = 2,\ z = -5-5t$. Not possible because the direction vector has 0 in the y-component.
8. $\vec{v} = \overrightarrow{PQ} = \langle 4,-2,3\rangle$, so $x = -4+4t,\ y = 3-2t,\ z = -5+3t$ is one possible answer (other answers are possible).
9. $2x-11y+z = 0$. 10. $31x-29y-13z = 0$.
11. $10x-2y+15z = 0$. We must solve for s from y, solve for r from z, then substitute these into x.
12. $Span(\{\langle 4,-10,6\rangle,\langle -6,15,-9\rangle\})$ is only a line through the origin, because the vectors are parallel to each other.
13. $x+y+z = 3$ 14. $9x+10y-2z = 28$

15. They determine a line because the vector $\overrightarrow{AB}$ is parallel to $\overrightarrow{AC}$.
17. $(3,-4,7)$ satisfies the equation. If $t = 1$, we get the point $(7,-7,13)$, which also satisfies the equation. Since two points on the line are also on the plane, the whole line is on the plane. Alternatively, you can solve for x, y and z from the equation of the line, and substitute them into that of the plane, and get $0 = 0$, showing that the equation of the plane is satisfied by every point on the line.
18. If $Q = (3,4,-1)$, then $\overrightarrow{PQ} = \langle 1,-1,-8\rangle$ is not parallel to $\langle 1,-2,5\rangle$, so P is not on L. Equation: $21x + 13y + z = 114$.
19. $7x + 2y + 4z = 15$ 20. $\left(-\frac{65}{29}, \frac{21}{29}, \frac{52}{29}\right)$
23. a. the point does not satisfy the symmetric equations; b. $\frac{x-5}{3} = \frac{y+2}{5} = -z+4$
24. $13x - 7y + 4z = 95$ 25. $17x - 4y + 22z = -80$
28. 28. a. $2x + 6y + 3z = 0$; b. $\vec{w}$ does not satisfy this equation.
33. d. $6x - 5y + 4z = 60$; g. $3x - 2z = 18$ i. $z = -5$
34. a. $D = (at + x_0 - x_1)^2 + (bt + y_0 - y_1)^2 + (ct + z_0 - z_1)^2$
    b. $\frac{dD}{dt} = 2[t + a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)]$
    c. $t = a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)$; d. $\frac{d^2D}{dt^2} = 2 > 0$.
    e. the critical point is a local minimum by the 2nd derivative test; since D goes to positive infinity in both directions, the critical point is also an absolute maximum.
35. The critical value is $t = \frac{-3}{\sqrt{66}}$; $\left(\frac{53}{11}, -\frac{67}{22}, \frac{51}{22}\right)$; distance: $\frac{7}{22}\sqrt{374}$
36. The critical value is $t = \frac{14}{\sqrt{30}}$; $\left(\frac{98}{15}, \frac{7}{3}, -\frac{46}{15}\right)$; distance: $\frac{1}{15}\sqrt{25530}$

1.3 Exercises

1. $\|\vec{u}\| = \sqrt{119}$.
2. 2. $\cos(\theta) = 5/\sqrt{3161}$ and $\theta \approx 1.481$ radians.
3. $\|2\vec{u} + 5\vec{v}\| = \sqrt{941} \approx 32.68$, and $\|2\vec{u}\| + \|5\vec{v}\| = \sqrt{136} + \sqrt{1625} \approx 51.97$. The second quantity should be bigger by the Triangle Inequality.
4. $\cos(\theta) = 37/\sqrt{6391}$, so $\theta \approx 1.09$ radians.
5. $\cos(\theta) = -15/\left(7\sqrt{23}\right)$, so $\theta \approx 2.034$ radians.
6. $\cos^{-1}\left(1/\sqrt{3}\right) = 54.7356^0$
7. 71.0682^0, 60.8784^0, 35.7958^0
8. -2911
9. $\sqrt{4569}$
10. $\sqrt{7837}$
11. 24
12. $\|\vec{u}\| = 29$, $\|\vec{v}\| = 13$, and $\|4\vec{u} + 9\vec{v}\| = \sqrt{2305}$
13. $6x - 5y + 2z = -15$.
14. $2x + 5y - 9z = 40$
15. Take the dot product with both $\vec{u}$ and $\vec{v}$.
16. a. $(13,3,6)$ b. $5x + 13y + z = 110$
17. $x + y - z = 10$; they intersect at $(2,5,-3)$.

18. $\langle x,y,z\rangle = \langle 5,-3,7\rangle + t\langle 9,22,17\rangle$; they intersect at $\left(\frac{97}{14}, \frac{12}{7}, \frac{149}{14}\right)$.
19. c. $7x + 5y - 3z = 50$
20. c. $7x + 11y - 13z = 46$ and $7x + 11y - 13z = 104$
21. $\langle 3,-5,2\rangle \circ \langle 2,4,7\rangle = 0$; $\langle x,y,z\rangle = \langle \frac{9}{22}, -\frac{21}{22}, 0\rangle + t\langle -43,-17,22\rangle$;
22. $4x + y - z = 20$
23. b. $15x + 13y + 10z = 68$; c. $\langle x,y,z\rangle = \langle 3,1,1\rangle + t\langle 2,0,-3\rangle$
24. The direction vector of L is a multiple of the normal vector to Π.
25. $8x + 5y - 4z = 2$; they intersect at $\left(\frac{118}{105}, \frac{62}{21}, \frac{571}{105}\right)$
26. $\langle x,y,z\rangle = \langle 5,-2,1\rangle + t\langle 3,7,-4\rangle$; they intersect at $\left(\frac{397}{74}, -\frac{85}{74}, \frac{19}{37}\right)$
27. b. $x + 2z = 12$.
28. False: the converse is True, but the forward implication is False; $\vec{u} \circ \vec{v} = 0$ means the two vectors are orthogonal to each other without one of them necessarily being $\vec{0}_n$.

1.4 Exercises

1. $\langle -3, 2, 6\rangle$; all variables are leading 2. $\langle -9, 4, 0\rangle$; all variables are leading
3. $\langle -3 - 7r, 2 + 4r, r\rangle$, $x_3 = r$ is free 4. $\langle 6 + 3r, r, -7\rangle$; $x_2 = r$ is free
5. $\langle 2, -5, r\rangle$; $x_3 = r$ is free 6. $\langle 8 + 5r - 2s, r, s\rangle$; $x_2 = r$ and $x_3 = s$ are free
7. $\langle 3 + 5r, -4r, -2 + 7r, r\rangle$; $x_4 = r$ is free 8. $\langle 5 - 3r, 6 + 2r, r, -4\rangle$; $x_3 = r$ is free
9. no solutions 10. $\langle 5 + 4r, r, -3 - s, s\rangle$; $x_2 = r$ and $x_4 = s$ are free
11. $\langle 7 + 2r - 6s, r, s, -2\rangle$; $x_2 = r$ and $x_3 = s$ are free 12. $\left\langle \frac{5}{3} + \frac{2}{3}r, -\frac{7}{3} - \frac{4}{3}r, \frac{2}{3} - \frac{1}{3}r, r\right\rangle$; $x_4 = r$ is free
13. $\langle -5, 3, 2\rangle$; all variables are leading 14. $\langle 2 - 3r, -4 + 5r, r\rangle$; $x_3 = r$ is free
15. $\langle 7 + 6r, r, -2\rangle$; $x_2 = r$ is free 16. $\langle 5, 6, -4, 0\rangle$; all variables are leading
17. $\langle 4r, 3 - 7r, -8 - 3r, r\rangle$; $x_4 = r$ is free 18. $\langle 1 + 6r, 5 - 4r, r, -4\rangle$; $x_3 = r$ is free
19. $\langle -2 + 5r, r, 3, 7\rangle$; $x_2 = r$ is free 20. $\langle -8 + 3r - 2s, -5 - 4r + 6s, r, s\rangle$; $x_3 = r$ and $x_4 = s$ are free
21. $\langle -2 + 5r + 9s, r, -6 - 4s, s\rangle$; $x_2 = r$ and $x_4 = s$ are free
22. $\langle -5 - 7r - 5s, 2 + 4r - 3s, 4 - 6r + 2s, r, s\rangle$; $x_4 = r$ and $x_5 = s$ are free
23. $\langle 5 - 3r + 4s + 6t, -1 + 2r + 9s - 8t, r, s, t\rangle$; $x_3 = r$, $x_4 = s$ and $x_5 = t$ are free.
24. $\langle -5 - 6r, 2 + 3r, 4 - 2r, -1 - 8r, r\rangle$; $x_5 = r$ is free 25. $\langle 5 - 3r, 6 + 2r, r, -4, 9\rangle$; $x_3 = r$ is free
26. $\langle 2 - 6r - 3s, r, 7 + 8s, s, -3\rangle$; $x_2 = r$ and $x_4 = s$ are free
27. $\langle -2 + 5r - 4s, r, 9 - 7s, 6 - 3s, s\rangle$; $x_2 = r$ and $x_5 = s$ are free
28. no solutions 29. $\langle r, 2 + 3s, s, -7, 4\rangle$; $x_1 = r$ and $x_3 = s$ are free
30. $\langle 4 + 5r - 3s, 5 - 3r, -2 + 2r - 4s, 3 - 7r + 6s, r, s\rangle$; $x_5 = r$ and $x_6 = s$ are free
31. $\langle 7 + 9r - 4s, -3r + s, r, -1 - 6s, 2 - 5s, s\rangle$; $x_3 = r$ and $x_6 = s$ are free
32. $\langle 2 - 6r - 3s - 5t, r, 9 + 8s + 2t, s, t, -1\rangle$, $x_2 = r$, $x_4 = s$ and $x_5 = t$ are free
33. $\langle 3 - 5r, -7 + 2r, r, 9, 4\rangle$, $x_3 = r$ is free 34. $\langle -2 + 4r - 7s, 5 - 6r + 3s, r, 6 - 9s, s\rangle$, $x_3 = r$, $x_5 = s$ are free
35. $\langle -2 + 8r + s, 6 - 5r - 7s, r, 3 + 4s, 8 - 9s, s\rangle$, $x_3 = r$, $x_6 = s$ are free
36. $\langle -5 - 6r, 2 + 7r, 3 - 4r, r, -8, 9\rangle$, $x_4 = r$ is free 37. Yes, $\vec{b} = 3\vec{v}_1 - 5\vec{v}_2$ (only solution)
38. $\vec{b}$ is not in $Span(S)$. 39. Yes. $\vec{b} = 3\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3$ (only solution)
40. Yes. $\vec{b} = \frac{1}{2}\vec{v}_1 + \frac{3}{2}\vec{v}_2$ (there are infinitely many solutions) 41. $\vec{b}$ is not in $Span(S)$.

42. Yes. $\vec{b} = 5\vec{v}_1 - 2\vec{v}_2 + 4\vec{v}_3$ (only solution) 43. Yes. $\vec{b} = -17\vec{v}_1 + 13\vec{v}_2$ (there are infinitely many solutions)

44. Yes. $\vec{b} = 3\vec{v}_1 - 2\vec{v}_2 + 5\vec{v}_3$ (there are infinitely many solutions)

45. Yes. $\vec{b} = -2\vec{v}_1 + 5\vec{v}_2$ (there are infinitely many solutions)

46. Yes. $\vec{b} = 2\vec{v}_1 - 7\vec{v}_2 + 3\vec{v}_4$ (there are infinitely many solutions) 47. Yes. $\vec{b} = \vec{v}_1 - \vec{v}_2 + 2\vec{v}_3$ (only solution)

48. Yes. $\vec{b} = 5\vec{v}_1 - 4\vec{v}_2$ (there are infinitely many solutions) 49. $\left\langle 0, \frac{2}{7}, -\frac{3}{7} \right\rangle$

50. $\left\langle \frac{43}{11}, -\frac{8}{11}, -\frac{8}{11}, \frac{2}{11} \right\rangle$ 51. $\left\langle -\frac{7}{5}, -\frac{8}{5}, -\frac{8}{5}, -\frac{7}{5} \right\rangle$ 52. $\left\langle -2s, 6s - \frac{47}{3}, \frac{8}{3}, s \right\rangle$, where $x_4 = s \in \mathbb{R}$.

53. $\langle -7, -1, -26, 31, 2, 7 \rangle$ 54. $\left\langle 8 - 6r - \frac{17t}{4}, r, -7, -2 - \frac{t}{4}, t, -1 \right\rangle$, where $x_5 = t \in \mathbb{R}$.

55. $\left\langle 8 - 9s, -\frac{1}{4} + \frac{25}{4}s, \frac{5 - 5s}{4}, 3 + 4s, 8 - 9s, s \right\rangle$, where $x_6 = s \in \mathbb{R}$. 56. $\langle 5, -3, -9 \rangle$ 57. $\langle 11, -3, 4 \rangle$

58. $\langle -14, -2, 3, 2 \rangle$ 59. $\langle -3 - 3r + 4s, r, -2 - 2s, s, 2 \rangle$, $x_2 = r \in \mathbb{R}, x_4 = s \in \mathbb{R}$ are free.

60. $\langle 3 + 5r, r, -2, 4 \rangle$, $y = r \in \mathbb{R}$ is free. 61. $\langle 3 - 5r, -7 + 2r, r, 4 \rangle$, $z = r \in \mathbb{R}$ is free.

62. No solutions. 63. One possible answer: $\langle x, y, z \rangle = \langle 40, 22, 0 \rangle + t\langle -43, -25, 2 \rangle$.

64. \$1.50 per shirt, \$5 per pair of slacks, and \$7 per jacket.

65. 1 kilogram of Barley, 3 kilograms of Oats, and 2 kilogram of Soy.

66. The rref is $\begin{bmatrix} 1 & 0 & -\frac{4}{5} & \frac{159}{5} \\ 0 & 1 & \frac{9}{5} & \frac{331}{5} \end{bmatrix}$, so $d = (159 + 4p)/5$ and $n = (331 - 9p)/5$.

The solution with the smallest number of pennies has $p = 4$, $n = 59$, and $d = 35$. (Note: since we want $n \geq 0$, we need $p \leq 36$) The solution with the largest number of pennies has $p = 34$, $n = 5$ and $d = 59$.

1.5 Exercises

1. a. consistent, and b. square
2. a. consistent, and b. overdetermined
3. a. inconsistent, and b. overdetermined
4. a. consistent, and b. underdetermined
5. a. inconsistent, and b. underdetermined
6. a. consistent, and b. square
7. a. consistent, and b. square
8. a. consistent, and b. underdetermined.
9. a. consistent, and b. square.
10. a. inconsistent, and b. overdetermined.
11. independent
12. independent
13. dependent
14. dependent
15. dependent
16. independent
17. dependent
18. dependent: $2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}_3$.

19. independent
20. independent
21. dependent: $2\vec{v}_1 - \vec{v}_2 + 5\vec{v}_3 = \vec{0}_4$.
22. dependent: $-4\vec{v}_1 - 7\vec{v}_2 + \vec{v}_3 = \vec{0}_4$.
23. dependent: $-3\vec{v}_1 - \vec{v}_2 + 5\vec{v}_3 = \vec{0}_5$.
24. dependent: $-2\vec{v}_1 - 3\vec{v}_2 + 4\vec{v}_3 + \vec{v}_4 = \vec{0}_5$
25. a. $-2\vec{v}_1 - 3\vec{v}_2 + \vec{v}_3 = \vec{0}_4$ b. $5\vec{v}_1 + 7\vec{v}_2 + \vec{v}_4 = \vec{0}_4$ c. $-\vec{v}_2 + 5\vec{v}_3 + 2\vec{v}_4 = \vec{0}_4$
26. a. $-2\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}_5$ b. $-3\vec{v}_1 - 2\vec{v}_2 + \vec{v}_4 = \vec{0}_5$ c. $-7\vec{v}_2 - 3\vec{v}_3 + 2\vec{v}_4 = \vec{0}_5$
27. a. $-3\vec{v}_1 - 5\vec{v}_2 + 6\vec{v}_3 + \vec{v}_4 = \vec{0}_4$ b. $-2\vec{v}_1 - 3\vec{v}_2 + 5\vec{v}_3 + \vec{v}_5 = \vec{0}_4$
    c. $-\vec{v}_1 + 7\vec{v}_3 - 3\vec{v}_4 + 5\vec{v}_5 = \vec{0}_4$
28. a. $-4\vec{v}_1 - 5\vec{v}_2 + \vec{v}_3 = \vec{0}_4$ b. $-3\vec{v}_1 - 2\vec{v}_2 + \vec{v}_4 + 2\vec{v}_5 = \vec{0}_4$
    c. $7\vec{v}_1 + 2\vec{v}_3 - 5\vec{v}_4 - 10\vec{v}_5 = \vec{0}_4$
29. a. $5\vec{v}_1 + 2\vec{v}_2 = \vec{0}_5$ b. $5\vec{v}_1 - 6\vec{v}_3 + 2\vec{v}_5 = \vec{0}_5$ c. $\vec{v}_3 + \vec{v}_4 + \vec{v}_5 = \vec{0}_5$
30. dependent: 5 vectors in $\mathbb{R}^4$ must be dependent.
31. One possible dependence equation is: $3(2\vec{u} + \vec{v}) - 1(4\vec{u} + 5\vec{v} - 4\vec{w}) - 2(\vec{u} - \vec{v} + 2\vec{w}) = \vec{0}_n$.
32. The system will have no solution if $r = -4$ and $s \neq \frac{7}{2}$. The system will have exactly one solution if $r \neq -4$ and s is ***any*** real number. The system will have an infinite number of solutions if $r = -4$ and $s = \frac{7}{2}$.
33. In all cases, x is a leading variable. The system will have no solution if $s = -8$ and $t \neq 4$. The system will have exactly one solution if $s \neq -8$, t is ***any*** real number, and $r \neq -6$. The system will have an infinite number of solutions involving exactly one free variable in two ways. First, if $s = -8$, $t = 4$, and $r \neq -6$, then y is a leading variable and z is a free variable. If $r = -6$, then z is automatically a leading variable because of the 2nd equation, and $z = -\frac{13}{10}$. This will satisfy the 3rd equation if and only if $(8+s)\left(-\frac{13}{10}\right) = t - 4$, so $10t + 13s = -144$. Thus, the second way is to have $r = -6$ and s and t any two real numbers satisfying $10t + 13s = -144$. In this case, y is a free variable. The system will never have an infinite number of solutions involving exactly two free variables.
34. $c = 22$
46. a. False. b. False. c. True. d. False e. True. f. False. g. True. h. False. i. True. j. False.

1.6 Exercises

1. The corresponding pairs of vectors are parallel to each other.
2. If we denote by $S = \{\vec{v}_1, \vec{v}_2\}$ and $S' = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, then we will get:
   $\vec{v}_1 = \frac{3}{5}\vec{w}_1 + \frac{1}{5}\vec{w}_2,\ \vec{v}_2 = \frac{1}{5}\vec{w}_1 - \frac{3}{5}\vec{w}_2,\ \vec{w}_1 = \frac{3}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2,\ \vec{w}_2 = \frac{1}{2}\vec{v}_1 - \frac{3}{2}\vec{v}_2,$
   $\vec{w}_3 = 2\vec{v}_1 - \vec{v}_2$.
3. We should apply the Equality of Spans Theorem; if $S = \{\vec{v}_1, \vec{v}_2\}$ and $S' = \{\vec{w}_1, \vec{w}_2\}$, then we will get:
   $\vec{v}_1 = \frac{1}{3}\vec{w}_1 + \frac{2}{3}\vec{w}_2,\ \vec{v}_2 = \frac{5}{3}\vec{w}_1 + \frac{16}{3}\vec{w}_2,\ \vec{w}_1 = 8\vec{v}_1 - \vec{v}_2,\ \vec{w}_2 = -\frac{5}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2.$
4. Although both Theorems are applicable, the first Theorem will certainly be easier to apply: corresponding pairs of vectors are parallel to each other.
5. a. S consists of 6 vectors from $\mathbb{R}^3$, so S is certainly dependent. b. $\vec{v}_2$ and $\vec{v}_4$ are parallel

to $\vec{v}_1$. c. Eliminate $\vec{v}_2$ and $\vec{v}_4$, to get: $S' = \{\vec{v}_1, \vec{v}_3, \vec{v}_5, \vec{v}_6\}$. You could also eliminate $\vec{v}_1$ and $\vec{v}_2$ and keep $\vec{v}_4$, or eliminate the $\vec{v}_1$ and $\vec{v}_4$ and keep $\vec{v}_2$. d. $\vec{v}_5 = \frac{5}{3}\vec{v}_1 + 2\vec{v}_3$. e. Eliminate either $\vec{v}_1$ or $\vec{v}_3$ or $\vec{v}_5$ to get a set with 3 vectors left. One possible answer is $S'' = \{\vec{v}_1, \vec{v}_3, \vec{v}_6\}$. f. The rref of the 3×3 matrix you obtained should not have any free variables.

6. a. S consists of 5 vectors from $\mathbb{R}^4$, so S is certainly dependent. b. $\vec{v}_4$ is parallel to $\vec{v}_2$. c. Eliminate either $\vec{v}_2$ or $\vec{v}_4$, so one possible answer is: $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_5\}$ d. $\vec{v}_3 = 3\vec{v}_1 - 2\vec{v}_2$ and $\vec{v}_5 = 2\vec{v}_1 + \vec{v}_2$. e. two vectors are left; one possible answer is: $S'' = \{\vec{v}_1, \vec{v}_2\}$. f. the two vectors (no matter which you picked) are obviously not parallel.
7. a, b and d only.
8. a, d and e only.
9. a, b, c, d and f only.
10. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$; $\vec{v}_4 = 3\vec{v}_1 + 2\vec{v}_2 - 4\vec{v}_3$.
11. $S' = \{\vec{v}_1, \vec{v}_2\}$; $\vec{v}_3 = 3\vec{v}_1 - 2\vec{v}_2$; $\vec{v}_4 = 2\vec{v}_1 + 3\vec{v}_2$.
12. $S' = \{\vec{v}_1, \vec{v}_3\}$; $\vec{v}_2 = -5\vec{v}_1$; $\vec{v}_4 = 3\vec{v}_1 + 5\vec{v}_3$.
13. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$; $\vec{v}_4 = 3\vec{v}_1 + 4\vec{v}_2 - 2\vec{v}_3$; $\vec{v}_5 = 2\vec{v}_1 + 3\vec{v}_2 - \vec{v}_3$.
14. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $\vec{v}_3 = 4\vec{v}_1 + 7\vec{v}_2$; $\vec{v}_5 = 3\vec{v}_1 + 4\vec{v}_2 - 2\vec{v}_4$.
15. $S' = \{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$; $\vec{v}_2 = -4\vec{v}_1$; $\vec{v}_5 = \frac{1}{2}\vec{v}_1 - \frac{3}{2}\vec{v}_3 + \frac{1}{2}\vec{v}_4$.
16. $S' = \{\vec{v}_1, \vec{v}_3, \vec{v}_5\}$; $\vec{v}_2 = 3\vec{v}_1$; $\vec{v}_4 = 4\vec{v}_1 + 2\vec{v}_3$.
17. $S' = \{\vec{v}_1, \vec{v}_2\}$; $\vec{v}_3 = 4\vec{v}_1 + 3\vec{v}_2$; $\vec{v}_4 = -\vec{v}_1 + 2\vec{v}_2$; $\vec{v}_5 = -2\vec{v}_1 - \vec{v}_2$.
18. $S' = \{\vec{v}_1, \vec{v}_2\}$; $\vec{v}_3 = 2\vec{v}_1 - 3\vec{v}_2$.
19. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.
20. $S' = \{\vec{v}_1, \vec{v}_2\}$; $\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$; $\vec{v}_4 = -6\vec{v}_1 + 5\vec{v}_2$.
21. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $\vec{v}_3 = 5\vec{v}_1 + 7\vec{v}_2$.
22. $S' = \{\vec{v}_1, \vec{v}_3\}$; $\vec{v}_2 = -3\vec{v}_1$; $\vec{v}_4 = 5\vec{v}_1 + 4\vec{v}_3$.
23. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $\vec{v}_3 = \frac{5}{2}\vec{v}_1 + \frac{9}{2}\vec{v}_2$; $\vec{v}_5 = \vec{v}_1 + 7\vec{v}_2 + 5\vec{v}_4$.
24. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_5\}$; $\vec{v}_4 = 5\vec{v}_1 + 4\vec{v}_2 - 2\vec{v}_3$.
25. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$; $\vec{v}_4 = 5\vec{v}_1 + 4\vec{v}_2 - 2\vec{v}_3$; $\vec{v}_5 = 7\vec{v}_1 + 5\vec{v}_2 - 4\vec{v}_3$.
26. $S' = \{\vec{v}_1, \vec{v}_2\}$; $\vec{v}_3 = \frac{1}{7}\vec{v}_1 - \frac{5}{7}\vec{v}_2$.
27. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.
28. $S' = \{\vec{v}_1, \vec{v}_2\}$; $\vec{v}_3 = \frac{4}{7}\vec{v}_1 + \frac{29}{7}\vec{v}_2$; $\vec{v}_4 = \frac{1}{7}\vec{v}_1 - \frac{5}{7}\vec{v}_2$.
29. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $\vec{v}_3 = 5\vec{v}_1 + 8\vec{v}_2$.
30. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$; $\vec{v}_4 = 2\vec{v}_1 - 3\vec{v}_2 - 4\vec{v}_3$.
31. $S' = \{\vec{v}_1, \vec{v}_3, \vec{v}_5\}$; $\vec{v}_2 = \frac{1}{6}\vec{v}_1$; $\vec{v}_4 = \frac{7}{6}\vec{v}_1 - 9\vec{v}_3$.
32. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $\vec{v}_3 = -6\vec{v}_1 + 5\vec{v}_2$; $\vec{v}_5 = 5\vec{v}_1 - 3\vec{v}_2$.
33. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5\}$; $\vec{v}_3 = 5\vec{v}_1 + 8\vec{v}_2$.
34. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $\vec{v}_3 = 7\vec{v}_1 - 9\vec{v}_2$; $\vec{v}_5 = 2\vec{v}_1 + \vec{v}_2 + 5\vec{v}_4$; $\vec{v}_6 = 4\vec{v}_1 - 6\vec{v}_2 - 3\vec{v}_4$.
35. $S' = \{\vec{v}_1, \vec{v}_3, \vec{v}_6\}$; $\vec{v}_2 = -4\vec{v}_1$; $\vec{v}_4 = (5/3)\vec{v}_1$; $\vec{v}_5 = (5/3)\vec{v}_1 + 2\vec{v}_3$.
36. $S' = \{\vec{v}_1, \vec{v}_2\}$; $\vec{v}_3 = 3\vec{v}_1 - 2\vec{v}_2$; $\vec{v}_4 = -5\vec{v}_2$; $\vec{v}_5 = 2\vec{v}_1 + \vec{v}_2$.
37. $S' = \{\vec{v}_1, \vec{v}_2\}$; $\vec{v}_3 = (-2/3)\vec{v}_1 + (7/3)\vec{v}_2$; $\vec{v}_4 = (1/3)\vec{v}_1 + (1/3)\vec{v}_2$; $\vec{v}_5 = (-1/3)\vec{v}_1 + (2/3)\vec{v}_2$.

38. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $\vec{v}_3 = -\dfrac{3}{2}\vec{v}_2$; $\vec{v}_5 = \dfrac{1}{2}\vec{v}_2 + \vec{v}_4$.
39. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $\vec{v}_3 = 2\vec{v}_1 - 4\vec{v}_2$; $\vec{v}_5 = 2\vec{v}_1 - 3\vec{v}_2$.
40. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5\}$; $\vec{v}_3 = 7\vec{v}_1 - 4\vec{v}_2$; $\vec{v}_6 = 6\vec{v}_1 - 7\vec{v}_2 + 3\vec{v}_4 - 5\vec{v}_5$.
41. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $\vec{v}_3 = 5\vec{v}_1 + 8\vec{v}_2$; $\vec{v}_5 = 5\vec{v}_1 + 7\vec{v}_2 + 4\vec{v}_4$; $\vec{v}_6 = 4\vec{v}_1 + 3\vec{v}_2 + 2\vec{v}_4$.
42. $S' = \{\vec{v}_1, \vec{v}_2\}$; $\vec{v}_3 = -2\vec{v}_1$; $\vec{v}_4 = -\vec{v}_1 + \vec{v}_2$; $\vec{v}_5 = -2\vec{v}_1 + 5\vec{v}_2$.
43. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $\vec{v}_3 = 4\vec{v}_1 + 3\vec{v}_2$; $\vec{v}_5 = 2\vec{v}_1 + 5\vec{v}_2 + 3\vec{v}_4$
44. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_5\}$; $\vec{v}_4 = 3\vec{v}_1 + 4\vec{v}_2 - 2\vec{v}_3$; $\vec{v}_6 = 4\vec{v}_1 + 2\vec{v}_2 - 3\vec{v}_3 - 5\vec{v}_5$.
45. $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_5, \vec{v}_6\}$; $\vec{v}_4 = 5\vec{v}_1 + 3\vec{v}_2 - 2\vec{v}_3$.
46. a. Two non-parallel vectors are independent. b. $x + 2z = 0$. c. only $\vec{e}_2$ is in $Span(S)$ d. Yes, because $\vec{e}_1 \notin Span(S)$. d. No, because $\vec{e}_2 \in Span(S)$. e. Yes, because $\vec{e}_3 \notin Span(S)$.

47. b. Yes. c. No. d. No. e. Yes.   48. b. $\begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & \frac{5}{2} & 0 & 0 & -\frac{37}{26} & \frac{27}{26} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & -\frac{7}{26} & \frac{3}{26} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{8}{13} & \frac{17}{13} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{6}{13} & \frac{16}{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{7}{13} & -\frac{16}{13} \end{bmatrix}$

c. independent d. dependent e. independent f. independent g. independent
53. a. False. b. True. c. False. d. True. e. False. f. False. g. False. h. False. i. False.

1.7 Exercises

1. $\{\langle 7,5\rangle\}$.
2. It doesn't contain the origin.
3. $\{\langle 7,3,0\rangle, \langle 0,4,7\rangle\}$ is one possibility (you can also use $\langle 4,0,-3\rangle$ as a second vector).
4. $\{\langle 5,0,2\rangle, \langle 0,1,0\rangle\}$
5. It doesn't contain the origin.
6. $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $dim(W) = 3$
7. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$; $dim(W) = 3$
8. $\{\vec{v}_1, \vec{v}_3, \vec{v}_6\}$; $dim(W) = 3$
9. $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $dim(W) = 3$
10. $\{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5\}$; $dim(W) = 4$
11. $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $dim(W) = 3$
12. $\{\vec{v}_1, \vec{v}_3, \vec{v}_6\}$; $dim(W) = 3$
13. $\{\vec{v}_1, \vec{v}_2\}$; $dim(W) = 2$
14. $\{\langle 5,-3,6,7\rangle, \langle 3,-1,4,5\rangle\}$; $dim(W) = 2$
15. $\{\langle 5,-3,6,7\rangle, \langle 3,-1,4,5\rangle, \langle 5,1,8,-3\rangle\}$; $dim(W) = 3$
16. $\{\langle 5,-3,6,7\rangle, \langle 3,-1,4,5\rangle, \langle 1,3,-1,1\rangle\}$; $dim(W) = 3$
17. $\{\langle 7,5,-4,3,9\rangle, \langle 4,3,-2,1,5\rangle\}$; $dim(W) = 2$
18. $\{\langle 7,5,-4,3,9\rangle, \langle 4,3,-2,1,5\rangle, \langle 4,3,-5,9,5\rangle\}$; $dim(W) = 3$
19. $\{\langle 7,5,-4,3,9\rangle, \langle 4,3,-2,1,5\rangle, \langle 4,3,-5,4,5\rangle\}$; $dim(W) = 3$
20. $\{\langle 5,-3,7,-4,6,3\rangle, \langle 9,-7,8,-9,4,7\rangle, \langle 4,-5,-3,-6,-7,5\rangle\}$; $dim(W) = 3$
21. $\{\langle 7,-3,4,2,-5,2\rangle, \langle 5,-2,3,3,-4,1\rangle, \langle -4,1,-3,-8,5,1\rangle\}$; $dim(W) = 3$

22. $\{\langle 7,-3,4,2,-5,2\rangle, \langle 5,-2,3,3,-4,1\rangle, \langle 6,-4,3,-9,-2,5\rangle, \langle -4,1,-3,-8,5,1\rangle\}$; $dim(W) = 4$
23. $\{\langle 7,-3,4,2,-5,2\rangle, \langle 5,-2,3,3,-4,1\rangle, \langle -4,1,-3,-2,4,-1\rangle\}$; $dim(W) = 3$
24. $\{\langle 7,-3,4,2,-5,2\rangle, \langle 5,-2,3,3,-4,1\rangle, \langle 8,-4,3,-9,-2,5\rangle, \langle -4,1,-3,-2,4,-1\rangle\}$; $dim(W) = 4$
25. $\{\langle 7,-3,4,2,-5,2\rangle, \langle 5,-2,3,3,-4,1\rangle, \langle 5,-3,2,-8,-1,4\rangle, \langle 8,-4,3,-9,-2,5\rangle, \langle -4,1,-3,-2,4$ $dim(W) = 5$
26. the xz-plane. $\{\langle 1,0,0\rangle, \langle 0,0,1\rangle\}$; $dim(W) = 2$
27. the x-axis. $\{\langle 1,0,0\rangle\}$; $dim(W) = 1$
28. W is not a subspace. It is not closed under addition.
29. $\{\langle 5,0,1,0\rangle, \langle 0,-1,0,1\rangle\}$; $dim(W) = 2$
30. $\{\langle 0,5,1,0,0\rangle, \langle 0,6,0,1,0\rangle, \langle -7,0,0,0,1\rangle\}$; $dim(W) = 3$
31. It does not contain the origin.
32. W is not a subspace, because it is not closed under scalar multiplication.

1.8 Exercises

1. $rowspace(A)$: $\{\langle 1,0,0,3\rangle, \langle 0,1,0,2\rangle, \langle 0,0,1,-4\rangle\}$; $colspace(A)$: $\{\langle 2,-3,4\rangle, \langle -3,0,-5\rangle, \langle 3,-1,-2\rangle\}$;
$nullspace(A)$: $\{\langle -3,-2,4,1\rangle\}$; $nullspace(A^\top) = \left\{\vec{0}_3\right\}$; $rank(A) = 3 = rank(A^\top)$;
$nullity(A) = 1$;
$nullity(A^\top) = 1$; $3 + 1 = 4$ and $3 + 0 = 3$;
$\langle 2,-3,3,-12\rangle = 2\langle 1,0,0,3\rangle - 3\langle 0,1,0,2\rangle + 3\langle 0,0,1,-4\rangle$
$\langle -3,0,-1,-5\rangle = -3\langle 1,0,0,3\rangle - \langle 0,0,1,-4\rangle$;
$\langle 4,-5,-2,10\rangle = 4\langle 1,0,0,3\rangle - 5\langle 0,1,0,2\rangle - 2\langle 0,0,1,-4\rangle$
2. $rowspace(A)$: $\{\langle 1,-5,0,3\rangle, \langle 0,0,1,5\rangle\}$; $colspace(A)$: $\{\langle -2,4,-3\rangle, \langle 3,-2,4\rangle\}$;
$nullspace(A)$: $\{\langle 5,1,0,0\rangle, \langle -3,0,-5,1\rangle\}$; $nullspace(A^\top)$: $\{\langle -10,1,8\rangle\}$;
$rank(A) = 2 = rank(A^\top)$;
$nullity(A) = 2$; $nullity(A^\top) = 1$; $2 + 2 = 4$ and $2 + 1 = 3$;
$\langle -2,10,3,9\rangle = -2\langle 1,-5,0,3\rangle + 3\langle 0,0,1,5\rangle$
$\langle 4,-20,-2,2\rangle = -4\langle 1,-5,0,3\rangle - 2\langle 0,0,1,5\rangle$; $\langle -3,15,4,11\rangle = -3\langle 1,-5,0,3\rangle + 4\langle 0,0,1,5\rangle$
3. $rowspace(A)$: $\{\langle 1,0,4,0,3\rangle, \langle 0,1,7,0,4\rangle, \langle 0,0,0,1,-2\rangle\}$; $colspace(A)$: $\{\langle 5,-2,3\rangle, \langle -2,3,-4\rangle, \langle -1,-3,2\rangle\}$;
$nullspace(A)$: $\{\langle -4,-7,1,0,0\rangle, \langle -3,-4,0,2,1\rangle\}$; $nullspace(A^\top) = \left\{\vec{0}_3\right\}$;
$rank(A) = 3 = rank(A^\top)$;
$nullity(A) = 2$; $nullity(A^\top) = 0$; $3 + 2 = 5$ and $3 + 0 = 3$;
$\langle 5,-2,6,-1,9\rangle = 5\langle 1,0,4,0,3\rangle - 2\langle 0,1,7,0,4\rangle - \langle 0,0,0,1,-2\rangle$
$\langle -2,3,13,-3,12\rangle = -2\langle 1,0,4,0,3\rangle + 3\langle 0,1,7,0,4\rangle - 3\langle 0,0,0,1,-2\rangle$
$\langle 3,-4,-16,2,-11\rangle = 3\langle 1,0,4,0,3\rangle - 4\langle 0,1,7,0,4\rangle + 2\langle 0,0,0,1,-2\rangle$
4. $rowspace(A)$: $\{\langle 1,3,0,4,0\rangle, \langle 0,0,1,2,0\rangle, \langle 0,0,0,0,1\rangle\}$; $colspace(A)$: $\{\langle -1,-3,2\rangle, \langle -2,3,-4\rangle, \langle 5,-2,3\rangle\}$;
$nullspace(A)$: $\{\langle -3,1,0,0,0\rangle, \langle -4,0,-2,1,0\rangle\}$; $nullspace(A^\top) = \left\{\vec{0}_3\right\}$;
$rank(A) = 3 = rank(A^\top)$;
$nullity(A) = 2$; $nullity(A^\top) = 0$; $3 + 2 = 5$ and $3 + 0 = 3$;
$\langle -1,-3,-2,-8,5\rangle = -\langle 1,3,0,4,0\rangle - 2\langle 0,0,1,2,0\rangle + 5\langle 0,0,0,0,1\rangle$
$\langle -3,-9,3,-6,-2\rangle = -3\langle 1,3,0,4,0\rangle + 3\langle 0,0,1,2,0\rangle - 2\langle 0,0,0,0,1\rangle$

$\langle 2,6,-4,0,3\rangle = 2\langle 1,3,0,4,0\rangle - 4\langle 0,0,1,2,0\rangle + 3\langle 0,0,0,0,1\rangle$

5. *rowspace*(A): $\{\langle 1,0,4,-1,-2\rangle, \langle 0,1,3,2,-1\rangle\}$; *colspace*$(A)$: $\{\langle -2,3,-5\rangle, \langle 5,-2,3\rangle\}$;
   nullspace(A): $\{\langle -4,-3,1,0,0\rangle, \langle 1,-2,0,1,0\rangle, \langle 2,1,0,0,1\rangle\}$; *nullspace*$(A^\top)$: $\{\langle 1,19,11\rangle\}$;
   rank$(A) = 2 =$ *rank*$(A^\top)$; *nullity*$(A) = 3$; *nullity*$(A^\top) = 1$; $2+3 = 5$ and $2+1 = 3$;
   $\langle -2,5,7,12,-1\rangle = -2\langle 1,0,4,-1,-2\rangle + 5\langle 0,1,3,2,-1\rangle$;
   $\langle 3,-2,6,-7,-4\rangle = 3\langle 1,0,4,-1,-2\rangle - 2\langle 0,1,3,2,-1\rangle$
   $\langle -5,3,-11,11,7\rangle = -5\langle 1,0,4,-1,-2\rangle + 3\langle 0,1,3,2,-1\rangle$
6. *rowspace*(A): $\{\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle\}$; *colspace*$(A)$:
   $\{\langle 3,7,1,-9\rangle, \langle -2,-4,0,-5\rangle, \langle 5,-6,8,2\rangle\}$;
   nullspace$(A) = \left\{\vec{0}_3\right\}$; *nullspace*$(A^\top)$: $\{\langle -14,243,141,200\rangle\}$;
   rank$(A) = 3 =$ *rank*$(A^\top)$;
   nullity$(A) = 0$; *nullity*$(A^\top) = 1$; $2+2 = 4$ and $2+2 = 4$;
   $\langle 3,-2,5\rangle = 3\langle 1,0,0\rangle - 2\langle 0,1,0\rangle + 5\langle 0,0,1\rangle$; $\langle 7,4,-6\rangle = 7\langle 1,0,0\rangle + 4\langle 0,1,0\rangle - 6\langle 0,0,1\rangle$
   $\langle 1,0,8\rangle = 1\langle 1,0,0\rangle + 8\langle 0,0,1\rangle$; $\langle -9,-5,2\rangle = -9\langle 1,0,0\rangle - 5\langle 0,1,0\rangle + 2\langle 0,0,1\rangle$
7. *rowspace*(A): $\{\langle 1,0,1,-6\rangle, \langle 0,1,2,5\rangle\}$; *colspace*$(A)$: $\{\langle 2,1,-2,-2\rangle, \langle 3,-2,1,-4\rangle\}$;
   nullspace(A): $\{\langle -1,-2,1,0\rangle, \langle 6,-5,0,1\rangle\}$; *nullspace*$(A^\top)$: $\{\langle 3,8,7,0\rangle, \langle 8,-2,0,7\rangle\}$;
   rank$(A) = 2 =$ *rank*$(A^\top)$; *nullity*$(A) = 2$; *nullity*$(A^\top) = 2$; $2+2 = 4$ and $2+2 = 4$;
   $\langle 2,3,8,3\rangle = 2\langle 1,0,1,-6\rangle + 3\langle 0,1,2,5\rangle$; $\langle 1,-2,-3,-16\rangle = \langle 1,0,1,-6\rangle - 2\langle 0,1,2,5\rangle$
   $\langle -2,1,0,17\rangle = -2\langle 1,0,1,-6\rangle + \langle 0,1,2,5\rangle$; $\langle -2,-4,-10,-8\rangle = -2\langle 1,0,1,-6\rangle - 4\langle 0,1,2,5\rangle$
8. *rowspace*(A): $\{\langle 1,-3,0,5\rangle, \langle 0,0,1,4\rangle\}$; *colspace*$(A)$: $\{\langle -3,7,5,4\rangle, \langle 1,-4,2,-3\rangle\}$;
   nullspace(A): $\{\langle 3,1,0,0\rangle, \langle -5,0,-4,1\rangle\}$; *nullspace*$(A^\top)$: $\{\langle 34,11,5,0\rangle, \langle -1,-1,0,1\rangle\}$;
   rank$(A) = 2 =$ *rank*$(A^\top)$; *nullity*$(A) = 2$; *nullity*$(A^\top) = 2$; $2+2 = 4$ and $2+2 = 4$;
   $\langle -3,9,1,-11\rangle = -3\langle 1,-3,0,5\rangle + \langle 0,0,1,4\rangle$; $\langle 7,-21,-4,19\rangle = 7\langle 1,-3,0,5\rangle - 4\langle 0,0,1,4\rangle$
   $\langle 5,-15,2,33\rangle = 5\langle 1,-3,0,5\rangle + 2\langle 0,0,1,4\rangle$; $\langle 4,-12,-3,8\rangle = 4\langle 1,-3,0,5\rangle - 3\langle 0,0,1,4\rangle$
9. *rowspace*(A): $\{\langle 2,0,5,0,2\rangle, \langle 0,2,9,0,14\rangle, \langle 0,0,0,1,5\rangle\}$; *colspace*$(A)$:
   $\{\langle 0,-7,8,-2\rangle, \langle 2,1,-2,-2\rangle, \langle -4,3,-1,6\rangle\}$;
   nullspace(A): $\{\langle -5,-9,2,0,0\rangle, \langle -1,-7,0,-5,1\rangle\}$; *nullspace*$(A^\top)$: $\{\langle 4,-6,-4,5\rangle\}$;
   rank$(A) = 3 =$ *rank*$(A^\top)$; *nullity*$(A) = 2$; *nullity*$(A^\top) = 1$; $3+2 = 5$ and $3+1 = 4$;
   $\langle 0,2,9,-4,-6\rangle = \langle 0,2,9,0,14\rangle - 4\langle 0,0,0,1,5\rangle$;
   $\langle -7,1,-13,3,15\rangle = -\frac{7}{2}\langle 2,0,5,0,2\rangle + \frac{1}{2}\langle 0,2,9,0,14\rangle + 3\langle 0,0,0,1,5\rangle$
   $\langle 8,-2,11,-1,-11\rangle = 4\langle 2,0,5,0,2\rangle - \langle 0,2,9,0,14\rangle - \langle 0,0,0,1,5\rangle$
   $\langle -2,-2,-14,6,14\rangle = -\langle 2,0,5,0,2\rangle - \langle 0,2,9,0,14\rangle + 6\langle 0,0,0,1,5\rangle$
10. *rowspace*(A): $\{\langle 1,0,0,5,7\rangle, \langle 0,1,0,4,5\rangle, \langle 0,0,1,-2,-4\rangle\}$; *colspace*$(A)$:
    $\{\langle 3,7,1,-9\rangle, \langle -2,-4,0,6\rangle, \langle 5,6,3,-9\rangle\}$;
    nullspace(A): $\{\langle -5,-4,2,1,0\rangle, \langle -7,-5,4,0,1\rangle\}$; *nullspace*$(A^\top)$: $\{\langle 9,6,-6,7\rangle\}$;
    rank$(A) = 3 =$ *rank*$(A^\top)$; *nullity*$(A) = 2$; *nullity*$(A^\top) = 1$; $3+2 = 5$ and $3+1 = 4$;
    $\langle 3,-2,5,-3,-9\rangle = 3\langle 1,0,0,5,7\rangle - 2\langle 0,1,0,4,5\rangle + 5\langle 0,0,1,-2,-4\rangle$
    $\langle 7,-4,6,7,5\rangle = 7\langle 1,0,0,5,7\rangle - 4\langle 0,1,0,4,5\rangle + 6\langle 0,0,1,-2,-4\rangle$
    $\langle 1,0,3,-1,-5\rangle = \langle 1,0,0,5,7\rangle + 3\langle 0,0,1,-2,-4\rangle$;
    $\langle -9,6,-9,-3,3\rangle = -9\langle 1,0,0,5,7\rangle + 6\langle 0,1,0,4,5\rangle - 9\langle 0,0,1,-2,-4\rangle$
11. *rowspace*(A): $\{\langle 7,0,4,1\rangle, \langle 0,7,29,-5\rangle\}$; *colspace*$(A)$:
    $\{\langle 15,-3,13,-9,-11\rangle, \langle 3,-2,4,1,2\rangle\}$;
    nullspace(A): $\{\langle -4,-29,7,0\rangle, \langle -1,5,0,7\rangle\}$; *nullspace*$(A^\top)$:
    $\{\langle -2,3,3,0,0\rangle, \langle 1,2,0,1,0\rangle, \langle 4,9,0,0,3\rangle\}$;
    rank$(A) = 2 =$ *rank*$(A^\top)$; *nullity*$(A) = 2$; *nullity*$(A^\top) = 3$; $2+2 = 4$ and $2+3 = 5$;

$\langle 15,3,21,0\rangle = \frac{15}{7}\langle 7,0,4,1\rangle + \frac{3}{7}\langle 0,7,29,-5\rangle$;
$\langle -3,-2,-10,1\rangle = \frac{-3}{7}\langle 7,0,4,1\rangle - \frac{2}{7}\langle 0,7,29,-5\rangle$
$\langle 13,4,24,-1\rangle = \frac{13}{7}\langle 7,0,4,1\rangle + \frac{4}{7}\langle 0,7,29,-5\rangle$;
$\langle -9,1,-1,-2\rangle = \frac{-9}{7}\langle 7,0,4,1\rangle + \frac{1}{7}\langle 0,7,29,-5\rangle$
$\langle -11,2,2,-3\rangle = \frac{-11}{7}\langle 7,0,4,1\rangle + \frac{2}{7}\langle 0,7,29,-5\rangle$

12. $rowspace(A)$: $\{\langle 1,0,5,0\rangle,\langle 0,1,8,0\rangle,\langle 0,0,0,1\rangle\}$; $colspace(A)$:
$(\{\langle 3,-2,-1,2\rangle,\langle 7,-4,3,6\rangle,\langle 1,0,5,1\rangle\})$;
$nullspace(A)$: $\{\langle -5,-8,1,0\rangle\}$; $nullspace(A^\top)$: $\{\langle -1,2,-2,1,0\rangle,\langle 23,-13,14,0,2\rangle\}$;
$rank(A) = 3 = rank(A^\top)$; $nullity(A) = 1$; $nullity(A^\top) = 2$; $3+1 = 4$ and $3+2 = 5$;
$\langle 3,-2,-1,2\rangle = 3\langle 1,0,5,0\rangle - 2\langle 0,1,8,0\rangle + 2\langle 0,0,0,1\rangle$;
$\langle 7,-4,3,6\rangle = 7\langle 1,0,5,0\rangle - 4\langle 0,1,8,0\rangle + 6\langle 0,0,0,1\rangle$
$\langle 1,0,5,1\rangle = \langle 1,0,5,0\rangle + \langle 0,0,0,1\rangle$;
$\langle -9,6,3,-8\rangle = -9\langle 1,0,5,0\rangle + 6\langle 0,1,8,0\rangle - 8\langle 0,0,0,1\rangle$
$\langle 4,-3,-4,9\rangle = 4\langle 1,0,5,0\rangle - 3\langle 0,1,8,0\rangle + 9\langle 0,0,0,1\rangle$

13. $rowspace(A)$: $\{\langle 1,0,0,2\rangle,\langle 0,1,0,-3\rangle,\langle 0,0,1,-4\rangle\}$; $colspace(A)$:
$\{\langle 5,-3,3,-9,-1\rangle,\langle 3,-2,4,1,2\rangle,\langle 2,-1,3,-1,2\rangle\}$;
$nullspace(A)$: $\{\langle -2,3,4,1\rangle\}$; $nullspace(A^\top)$: $\{\langle 30,29,-9,4\rangle,\langle 2,-3,-5,0,4\rangle\}$;
$rank(A) = 3 = rank(A^\top)$; $nullity(A) = 1$; $nullity(A^\top) = 2$; $3+1 = 4$ and $3+2 = 5$;
$\langle 5,3,2,-7\rangle = 5\langle 1,0,0,2\rangle + 3\langle 0,1,0,-3\rangle + 2\langle 0,0,1,-4\rangle$;
$\langle -3,-2,-1,4\rangle = -3\langle 1,0,0,2\rangle - 2\langle 0,1,0,-3\rangle - \langle 0,0,1,-4\rangle$
$\langle 3,4,3,-18\rangle = 3\langle 1,0,0,2\rangle + 4\langle 0,1,0,-3\rangle + 3\langle 0,0,1,-4\rangle$;
$\langle -9,1,-1,-17\rangle = -9\langle 1,0,0,2\rangle + \langle 0,1,0,-3\rangle - \langle 0,0,1,-4\rangle$
$\langle -1,2,2,-16\rangle = -1\langle 1,0,0,2\rangle + 2\langle 0,1,0,-3\rangle + 2\langle 0,0,1,-4\rangle$

14. $rowspace(A)$: $\{\langle 6,1,0,7,0\rangle,\langle 0,0,1,-9,0\rangle,\langle 0,0,0,0,1\rangle\}$;
$colspace(A)$: $\{\langle 12,-6,18,-6,12\rangle,\langle 3,-2,4,1,2\rangle,\langle 5,-3,0,7,-1\rangle\}$;
$nullspace(A)$: $\{\langle -1,6,0,0,0\rangle,\langle -7,0,54,6,0\rangle\}$;
$nullspace(A^\top) = Span(\{\langle 1,4,1,1,0\rangle,\langle -4,-9,-5,0,7\rangle\})$;
$rank(A) = 3 = rank(A^\top)$; $nullity(A) = 2$; $nullity(A^\top) = 2$; $3+2 = 5$ and $3+2 = 5$;
$\langle 12,2,3,-13,5\rangle = 2\langle 6,1,0,7,0\rangle + 3\langle 0,0,1,-9,0\rangle + 5\langle 0,0,0,0,1\rangle$
$\langle -6,-1,-2,11,-3\rangle = -\langle 6,1,0,7,0\rangle - 2\langle 0,0,1,-9,0\rangle - 3\langle 0,0,0,0,1\rangle$
$\langle 18,3,4,-15,0\rangle = 3\langle 6,1,0,7,0\rangle + 4\langle 0,0,1,-9,0\rangle$;
$\langle -6,-1,1,-16,7\rangle = -6\langle 6,1,0,7,0\rangle + \langle 0,0,1,-9,0\rangle + 7\langle 0,0,0,0,1\rangle$
$\langle 12,2,2,-4,-1\rangle = 2\left\langle 1,\frac{1}{6},0,\frac{7}{6},0\right\rangle + 2\langle 0,0,1,-9,0\rangle - \langle 0,0,0,0,1\rangle$

15. $rowspace(A)$: $\{\langle 1,0,5,0,0\rangle,\langle 0,1,8,0,0\rangle,\langle 0,0,0,1,0\rangle,\langle 0,0,0,0,1\rangle\}$;
$colspace(A)$: $\{\langle 3,7,1,-9,4\rangle,\langle -2,-4,0,6,-3\rangle,\langle 2,6,1,-8,9\rangle,\langle 5,6,3,-9,7\rangle\}$;
$nullspace(A)$: $\{\langle -5,-8,1,0,0\rangle\}$; $nullspace(A^\top)$: $\{\langle 91,82,-74,93,16\rangle\}$;
$rank(A) = 4 = rank(A^\top)$; $nullity(A) = 1 = nullity(A^\top)$; $4+1 = 5$ for both matrices;
$\langle 3,-2,-1,2,5\rangle = 3\langle 1,0,5,0,0\rangle - 2\langle 0,1,8,0,0\rangle + 2\langle 0,0,0,1,0\rangle + 5\langle 0,0,0,0,1\rangle$
$\langle 7,-4,3,6,6\rangle = 7\langle 1,0,5,0,0\rangle - 4\langle 0,1,8,0,0\rangle + 6\langle 0,0,0,1,0\rangle + 6\langle 0,0,0,0,1\rangle$
$\langle 1,0,5,1,3\rangle = \langle 1,0,5,0,0\rangle + 5\langle 0,0,0,1,0\rangle + 3\langle 0,0,0,0,1\rangle$
$\langle -9,6,3,-8,-9\rangle = 9\langle 1,0,5,0,0\rangle - 6\langle 0,1,8,0,0\rangle - 8\langle 0,0,0,1,0\rangle - 9\langle 0,0,0,0,1\rangle$
$\langle 4,-3,-4,9,7\rangle = 4\langle 1,0,5,0,0\rangle - 3\langle 0,1,8,0,0\rangle + 9\langle 0,0,0,1,0\rangle + 7\langle 0,0,0,0,1\rangle$

16. $rowspace(A)$: $\{\langle 1,0,7,0,2,4\rangle,\langle 0,1,-9,0,1,-6\rangle,\langle 0,0,0,1,5,-3\rangle\}$;
$colspace(A)$: $\{\langle 2,-1,3,-1,2\rangle,\langle 3,-2,4,1,2\rangle,\langle 1,-3,2,-2,-1\rangle\}$;

$nullspace(A)$: $\{\langle -7,9,1,0,0,0\rangle, \langle -2,-1,0,-5,1,0\rangle, \langle -4,6,0,3,0,1\rangle\}$;
$nullspace(A^{\top})$: $\{\langle -19,1,14,3,0\rangle, \langle 0,-1,-1,0,1\rangle\}$; $rank(A) = 3 = rank(A^{\top})$;
$nullity(A) = 3$;
$nullity(A^{\top}) = 2$; $3 + 3 = 6$ and $3 + 2 = 5$;
$\langle 2,3,-13,1,12,-13\rangle = 2\langle 1,0,7,0,2,4\rangle + 3\langle 0,1,-9,0,1,-6\rangle + \langle 0,0,0,1,5,-3\rangle$
$\langle -1,-2,11,-3,-19,17\rangle = -1\langle 1,0,7,0,2,4\rangle - 2\langle 0,1,-9,0,1,-6\rangle - 3\langle 0,0,0,1,5,-3\rangle$
$\langle 3,4,-15,2,20,-18\rangle = 3\langle 1,0,7,0,2,4\rangle + 4\langle 0,1,-9,0,1,-6\rangle + 2\langle 0,0,0,1,5,-3\rangle$
$\langle -1,1,-16,-2,-11,-4\rangle = -\langle 1,0,7,0,2,4\rangle + \langle 0,1,-9,0,1,-6\rangle - 2\langle 0,0,0,1,5,-3\rangle$
$\langle 2,2,-4,-1,1,-1\rangle = 2\langle 1,0,7,0,2,4\rangle + \langle 20,1,-9,0,1,-6\rangle - \langle 0,0,0,1,5,-3\rangle$

17. $\langle 3, 0, -2, 0\rangle + x_4\langle 5, -4, 7, 1\rangle$
18. $\langle 5, 6, 0, -4\rangle + x_3\langle -3, 2, 1, 0\rangle$
19. $\langle 5, 0, -3, 0\rangle + x_2\langle 4, 1, 0, 0\rangle + x_4\langle 0, 0, -1, 1\rangle$
20. $\langle 7, 0, 0, -2\rangle + x_2\langle 2, 1, 0, 0\rangle + x_3\langle -6, 0, 1, 0\rangle$
21. $\langle 2, -4, 0\rangle + x_3\langle -3, 5, 1\rangle$
22. $\langle 7, 0, -2\rangle + x_2\langle 6, 1, 0\rangle$
23. $\langle 0, 3, -8, 0\rangle + x_4\langle 4, -7, -3, 1\rangle$
24. $\langle -2, 0, 3, 7\rangle + x_2\langle 5, 1, 0, 0\rangle$
25. $\langle -8, -5, 0, 0\rangle + x_3\langle 3, -4, 1, 0\rangle + x_4\langle -2, 6, 0, 1\rangle$
26. $\langle -5, 2, 4, 0, 0\rangle + x_4\langle -7, 4, -6, 1, 0\rangle + x_5\langle -5, 3, 2, 0, 1\rangle$
27. $\langle 5, -1, 0, 0, 0\rangle + x_3\langle -3, 2, 1, 0, 0\rangle + x_4\langle 4, 9, 0, 1, 0\rangle + x_5\langle 6, -8, 0, 0, 1\rangle$
28. $\langle 5, 6, 0, -4, 9\rangle + x_3\langle -3, 2, 1, 0, 0\rangle$
29. $\langle -2, 0, 9, 6, 0\rangle + x_2\langle 5, 1, 0, 0, 0\rangle + x_5\langle -4, 0, -7, -3, 1\rangle$
30. $\langle 0, 2, 0, -7, 4\rangle + x_1\langle 1, 0, 0, 0, 0\rangle + x_3\langle 0, 3, 1, 0, 0\rangle$
31. $\langle 4, 5, -2, 3, 0, 0\rangle + x_5\langle 5, -3, 2, -7, 1, 0\rangle + x_6\langle -3, 0, -4, 6, 0, 1\rangle$
32. $\langle 2, 0, 9, 0, 0, -1\rangle + x_2\langle -6, 1, 0, 0, 0, 0\rangle + x_4\langle -3, 0, 8, 1, 0, 0\rangle + x_5\langle -5, 0, 2, 0, 1, 0\rangle$
33. $\langle 3, -7, 0, 9, 4\rangle + x_3\langle -5, 2, 1, 0, 0\rangle$
34. $\langle -2, 5, 0, 6, 0\rangle + x_3\langle 4, -6, 1, 0, 0\rangle + x_5\langle -7, 3, 0, -9, 1\rangle$
35. $\langle -2, 6, 0, 3, 8, 0\rangle + x_3\langle 8, -5, 1, 0, 0, 0\rangle + x_6\langle 1, -7, 0, 4, -9, 1\rangle$
36. $\langle -5, 2, 3, 0, -8, 9\rangle + x_4\langle -6, 7, -4, 1, 0, 0\rangle$
37. $\langle 7, -8, 0\rangle + x_3\langle 4, -5, 1\rangle$
38. $\langle 7, 6, -4, 0\rangle + x_4\langle -3, 2, 5, 1\rangle$
39. $\langle 4, -3, 0, 9\rangle + x_3\langle 2, -7, 1, 0\rangle$
40. $\langle 3, -2, 0, 0\rangle + x_2\langle 4, 1, 0, 0\rangle + x_4\langle -5, 0, -7, 1\rangle$
41. $\langle -3, 2, 0, 0, 0\rangle + x_3\langle -4, 7, 1, 0, 0\rangle + x_4\langle 9, -3, 0, 1, 0\rangle + x_5\langle -6, 5, 0, 0, 1\rangle$
42. $\langle 5, 0, 0, 4, 0\rangle + x_2\langle -4, 1, 0, 0, 0\rangle + x_3\langle 6, 0, 1, 0, 0\rangle + x_5\langle -7, 0, 0, 3, 1\rangle$
43. $\langle 6, 0, -11, 0, 0\rangle + x_2\langle 5, 1, 0, 0, 0\rangle + x_4\langle -4, 0, 2, 1, 0\rangle + x_5\langle 7, 0, -4, 0, 1\rangle$
44. $\langle -5, 3, -4, 2, 0\rangle + x_5\langle -3, 5, 2, -7, 1\rangle$
45. $\langle 3, 8, 0, -2, 7, 0\rangle + x_3\langle -5, 3, 1, 0, 0, 0\rangle + x_6\langle 1, 0, 0, -4, 6, 1\rangle$
46. $rowspace(A)$: $\{\langle 1, 0, 4, 5\rangle, \langle 0, 1, -2, -3\rangle\}$; $colspace(A)$: $\{\langle 3, 5, 16\rangle, \langle 2, 7, 29\rangle\}$;
$nullspace(A)$: $\{\langle -4, 2, 1, 0\rangle, \langle -5, 3, 0, 1\rangle\}$; $nullspace(A^{\top})$: $\{\langle 3, -5, 1\rangle\}$;
$rank(A) = 2 = rank(A^{\top})$; $nullity(A) = 2$; $nullity(A^{\top}) = 1$; $2 + 2 = 4$ and $2 + 1 = 3$.
47. $rowspace(A)$: $\{\langle 1, 0, 4, 0\rangle, \langle 0, 1, -3, 0\rangle, \langle 0, 0, 0, 1\rangle\}$; $colspace(A)$:
$\{\langle 5, -4, 3\rangle, \langle 6, -7, 2\rangle, \langle 1, 2, 3\rangle\}$;
$nullspace(A)$: $\{\langle -4, 3, 1, 0\rangle\}$; $nullspace(A^{\top}) = \{\vec{0}_3\}$;

$rank(A) = 3 = rank(A^\top)$; $nullity(A) = 1$; $nullity(A^\top) = 0$; $3 + 1 = 4$; $3 + 0 = 3$.

48. $rowspace(A)$: $\{\langle 1, 0, 4, 0, 6\rangle, \langle 0, 1, -3, 0, -3\rangle, \langle 0, 0, 0, 1, -4\rangle\}$;
$colspace(A)$: $\{\langle 5, 4, 3\rangle, \langle 6, 7, 2\rangle, \langle 1, -2, 3\rangle\}$; $nullspace(A)$:
$\{\langle -4, 3, 1, 0, 0\rangle, \langle -6, 3, 0, 4, 1\rangle\}$;
$nullspace(A^\top) = \left\{\vec{0}_3\right\}$; $rank(A) = 3 = rank(A^\top)$; $nullity(A) = 2$; $nullity(A^\top) = 0$;
$3 + 2 = 5$; $3 + 0 = 3$.

49. $rowspace(A)$: $\{\langle 1, 0, 4, 0, 2\rangle, \langle 0, 1, -3, 0, -5\rangle, \langle 0, 0, 0, 1, 7\rangle\}$;
$colspace(A)$: $\{\langle 3, 5, 1, 4\rangle, \langle 4, 7, 2, 3\rangle, \langle 3, 4, -1, 2\rangle\}$;
$nullspace(A)$: $\{\langle -4, 3, 1, 0, 0\rangle, \langle -2, 5, 0, -7, 1\rangle\}$; $nullspace(A^\top)$: $\{\langle 3, -2, 1, 0\rangle\}$;
$rank(A) = 3 = rank(A^\top)$; $nullity(A) = 2$; $nullity(A^\top) = 1$; $3 + 2 = 5$; $3 + 1 = 4$.

50. $rowspace(A)$: $\{\langle 1, 0, 5, 0, -8\rangle, \langle 0, 1, -7, 0, 3\rangle, \langle 0, 0, 0, 1, 7\rangle\}$;
$colspace(A)$: $\{\langle 4, 6, 17, 28\rangle, \langle 2, 3, 8, 13\rangle, \langle 5, 7, 20, 30\rangle\}$;
$nullspace(A)$: $\{\langle -5, 7, 1, 0, 0\rangle, \langle 8, -3, 0, -7, 1\rangle\}$; $nullspace(A^\top)$: $\{\langle 9, -5, -2, 1\rangle\}$;
$rank(A) = 3 = rank(A^\top)$; $nullity(A) = 2$; $nullity(A^\top) = 1$; $3 + 2 = 5$; $3 + 1 = 4$.

51. $rowspace(A)$: $\{\langle 1, 0, -7, 0, -9\rangle, \langle 0, 1, 4, 0, 3\rangle, \langle 0, 0, 0, 1, 2\rangle\}$;
$colspace(A)$: $\{\langle 4, 2, 5, 7, 10\rangle, \langle 11, 5, 12, 9, 19\rangle, \langle 9, 4, 10, 8, 17\rangle\}$;
$nullspace(A)$: $\{\langle 7, -4, 1, 0, 0\rangle, \langle 9, -3, 0, -2, 1\rangle\}$; $nullspace(A^\top)$:
$\{\langle 6, -3, -5, 1, 0\rangle, \langle 3, 4, -6, 0, 1\rangle\}$;
$rank(A) = 3 = rank(A^\top)$; $nullity(A) = 2 = nullity(A^\top)$; $3 + 2 = 5$ for both A and $A^\top$.

52. $rowspace(A)$: $\{\langle 1, 0, 4, 0, 0\rangle, \langle 0, 1, -5, 0, 0\rangle, \langle 0, 0, 0, 1, 0\rangle, \langle 0, 0, 0, 0, 1\rangle\}$;
$colspace(A)$: $\{\langle 3, 1, 0, -1, -4\rangle, \langle 2, 1, 2, -6, -7\rangle, \langle -1, -1, -3, 7, 8\rangle, \langle -1, 0, 4, -13, -6\rangle\}$;
$nullspace(A)$: $\{\langle -4, 5, 1, 0, 0\rangle\}$; $nullspace(A^\top)$: $\{\langle 3, -8, 4, 1, 0\rangle\}$;
$rank(A) = 4 = rank(A^\top)$; $nullity(A) = 1 = nullity(A^\top)$; $4 + 1 = 5$ for both A and $A^\top$.

53. $rowspace(A)$: $\{\langle 1, 0, -2, -3, 0\rangle, \langle 0, 1, 6, 5, 0\rangle, \langle 0, 0, 0, 0, 1\rangle\}$;
$colspace(A)$: $\{\langle 4, 9, 0, 11, -6, -9\rangle, \langle 2, 4, 2, 5, -2, -4\rangle, \langle 1, 2, 1, 2, 1, 1\rangle\}$;
$nullspace(A)$: $\{\langle 2, -6, 1, 0, 0\rangle, \langle 3, -5, 0, 1, 0\rangle\}$;
$nullspace(A^\top)$: $\{\langle -9, 4, 1, 0, 0, 0\rangle, \langle -5, -2, 0, 4, 1, 0\rangle, \langle -3, -5, 0, 6, 0, 1\rangle\}$;
$rank(A) = 3 = rank(A^\top)$; $nullity(A) = 2$; $nullity(A^\top) = 3$; $3 + 2 = 5$, and $3 + 3 = 6$.

54. $rowspace(A)$: $\{\langle 1, 0, 1, 0, 0\rangle, \langle 0, 1, -7, 0, 0\rangle, \langle 0, 0, 0, 1, 0\rangle, \langle 0, 0, 0, 0, 1\rangle\}$;
$colspace(A)$:
$\{\langle 3, 0, 12, -1, 12, -1\rangle, \langle 1, -1, 1, -1, 0, 0\rangle, \langle -2, -2, -14, -1, -17, 4\rangle, \langle 0, 5, 15, 4, 22, -6\rangle\}$;
$nullspace(A)$: $\{\langle -1, 7, 1, 0, 0\rangle\}$; $nullspace(A^\top)$:
$\{\langle -4, -3, 1, 0, 0, 0\rangle, \langle -5, -2, 0, -3, 1, 0\rangle\}$;
$rank(A) = 4 = rank(A^\top)$; $nullity(A) = 1$; $nullity(A^\top) = 2$; $4 + 1 = 5$, and $4 + 2 = 6$.

55. $rowspace(A)$: $\{\langle 1, 0, 2, 0, 0, 5\rangle, \langle 0, 1, 3, 0, 0, 2\rangle, \langle 0, 0, 0, 1, 0, 7\rangle, \langle 0, 0, 0, 0, 1, 4\rangle\}$;
$colspace(A)$:
$\{\langle 3, 4, 1, -6, -1, 9\rangle, \langle 1, -3, -2, 1, 1, 2\rangle, \langle 0, 2, 1, -2, 2, -18\rangle, \langle -4, -5, -1, 9, -3, 18\rangle\}$;
$nullspace(A)$: $\{\langle -2, -3, 1, 0, 0, 0\rangle, \langle -5, -2, 0, -7, -4, 1\rangle\}$; $nullspace(A^\top)$:
$\{\langle -2, 5, -8, 1, 0, 0\rangle, \langle -4, 3, -2, 0, 7, 1\rangle\}$;
$rank(A) = 4 = rank(A^\top)$; $nullity(A) = 2 = nullity(A^\top)$; $4 + 2 = 6$ for both A and $A^\top$.

56. $rowspace(A)$:
$\{\langle 1, 0, 4, 5, 0, 0, 3\rangle, \langle 0, 1, 9, 8, 0, 0, -4\rangle, \langle 0, 0, 0, 0, 1, 0, -4\rangle, \langle 0, 0, 0, 0, 0, 1, 5\rangle\}$;
$colspace(A)$: $\{\langle 3, -4, -3, 1, -5\rangle, \langle -1, 3, 2, 4, 3\rangle, \langle 2, 2, 1, 12, 0\rangle, \langle 2, 3, 1, 17, 1\rangle\}$;
$nullspace(A)$: $\{\langle -4, -9, 1, 0, 0, 0, 0\rangle, \langle -5, -8, 0, 1, 0, 0, 0\rangle, \langle -3, 4, 0, 0, 4, -5, 1\rangle\}$;
$nullspace(A^\top)$: $\{\langle -3, -5, 4, 1, 0\rangle\}$; $rank(A) = 4 = rank(A^\top)$; $nullity(A) = 3$;
$nullity(A^\top) = 1$;

$4+3=7$, and $4+1=5$.

63. 6×13. 67. a. False b. True c. False d. False e. True f. True g. False h. True i. False j. False k. True l. True m. False n. False o. False.

1.9 Exercises

1. $\begin{bmatrix} 1 & 0 & \frac{43}{11} \\ 0 & 1 & -\frac{13}{11} \end{bmatrix}$; W: $\{\langle 11,0,43\rangle,\langle 0,11,-13\rangle\}$; $W^{\perp}$: $\{\langle -43,13,11\rangle\}$; $dim(W)=2$; $dim(W^{\perp})=1$; $2+1=3$.

2. $\begin{bmatrix} 1 & 0 & \frac{26}{17} & -\frac{1}{17} \\ 0 & 1 & \frac{11}{34} & -\frac{11}{17} \end{bmatrix}$; W: $\{\langle 17,0,26,-1\rangle,\langle 0,34,11,-22\rangle\}$; $W^{\perp}$: $\{\langle -52,-11,34,0\rangle,\langle 1,11,0,17\rangle\}$; $dim(W)=2$; $dim(W^{\perp})=2$; $2+2=4$.

3. $\begin{bmatrix} 1 & 0 & \frac{5}{11} & \frac{1}{11} & \frac{11}{11} \\ 0 & 1 & -\frac{20}{11} & -\frac{37}{11} & \frac{18}{11} \end{bmatrix}$; W: $\{\langle 11,0,5,1,12\rangle,\langle 0,11,-20,-37,18\rangle\}$; $W^{\perp}$: $\{\langle -5,20,11,0,0\rangle,\langle -1,37,0,11,0\rangle,\langle -12,-18,0,0,11\rangle\}$; $dim(W)=2$; $dim(W^{\perp})=3$; $2+3=5$.

4. $\begin{bmatrix} 1 & -\frac{5}{2} & 3 & -\frac{3}{2} \end{bmatrix}$; W: $\{\langle 2,-5,6,-3\rangle\}$; $W^{\perp}$: $\{\langle 5,2,0,0\rangle,\langle -3,0,1,0\rangle,\langle 3,0,0,2\rangle\}$; $dim(W)=1$; $dim(W^{\perp})=3$; $1+3=4$.

5. $\begin{bmatrix} 1 & -\frac{1}{3} & \frac{5}{3} & \frac{2}{3} & 2 \end{bmatrix}$; W: $\{\langle 3,-1,5,2,6\rangle\}$; $W^{\perp}$: $\{\langle 1,3,0,0,0\rangle,\langle -5,0,3,0,0\rangle,\langle -2,0,0,3,0\rangle,\langle -2,0,0,0,1\rangle\}$; $dim(W)=1$; $dim(W^{\perp})=4$; $1+4=5$.

6. $\begin{bmatrix} 1 & 0 & \frac{26}{17} & 0 \\ 0 & 1 & \frac{11}{34} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; W: $\{\langle 17,0,26,0\rangle,\langle 0,34,11,0\rangle,\langle 0,0,0,1\rangle\}$; $W^{\perp}$: $\{\langle -52,-11,34,0\rangle\}$; $dim(W)=3$; $dim(W^{\perp})=1$; $3+1=4$.

7. $\begin{bmatrix} 1 & 0 & \frac{26}{17} & -\frac{1}{17} \\ 0 & 1 & \frac{11}{34} & -\frac{11}{17} \\ 0 & 0 & 0 & 0 \end{bmatrix}$; W: $\{\langle 17,0,26,-1\rangle,\langle 0,34,11,-22\rangle\}$; $W^{\perp}$: $\{\langle -52,-11,34,0\rangle,\langle 1,11,0,17\rangle\}$; $dim(W)=2$; $dim(W^{\perp})=2$; $2+2=4$.

8. $\begin{bmatrix} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & 3 & -5 \\ 0 & 0 & 1 & -4 & 7 \end{bmatrix}$; W: $\{\langle 1,0,0,-2,3\rangle,\langle 0,1,0,3,-5\rangle,\langle 0,0,1,-4,7\rangle\}$; $W^{\perp}$: $\{\langle 2,-3,4,1,0\rangle,\langle -3,5,-7,0,1\rangle\}$; $dim(W)=3$; $dim(W^{\perp})=2$; $3+2=5$.

9. $\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{17}{16} \\ 0 & 1 & 0 & 0 & -\frac{1}{8} \\ 0 & 0 & 1 & 0 & \frac{3}{16} \\ 0 & 0 & 0 & 1 & -\frac{5}{8} \end{bmatrix}$; W: $\{\langle 16,0,0,0,17\rangle, \langle 0,8,0,0,-1\rangle, \langle 0,0,16,0,3\rangle, \langle 0,0,0,8,-5\rangle\}$;
$W^{\perp}$: $\{\langle -17,2,-3,10,16\rangle\}$; $dim(W) = 4$; $dim(W^{\perp}) = 1$; $4+1 = 5$.

10. $\begin{bmatrix} 1 & 0 & 0 & -17 & 9 \\ 0 & 1 & 0 & -29 & 16 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; $W = Span(\{\langle 1,0,0,-17,9\rangle, \langle 0,1,0,-29,16\rangle, \langle 0,0,1,-2,2\rangle\})$;
$W^{\perp} = Span(\{\langle 17,29,2,1,0\rangle, \langle -9,-16,-2,0,1\rangle\})$; $dim(W) = 3$; $dim(W^{\perp}) = 2$; $3+2 = 5$.

11. a. Yes. b. Yes. c. No. d. Yes. e. No. f. No.

12. a. $R = \begin{bmatrix} 1 & 0 & \frac{5}{4} & -\frac{3}{4} & -\frac{11}{4} \\ 0 & 1 & -\frac{9}{8} & -\frac{13}{8} & -\frac{17}{8} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; b. $\{\langle -10,9,8,0,0\rangle, \langle 6,13,0,8,0\rangle, \langle 22,17,0,0,8\rangle\}$

c. $\{\langle 4,0,5,-3,-11\rangle, \langle 0,8,-9,-13,-17\rangle\}$ d. $dim(W) = 2$; $dim(W^{\perp}) = 3$; $2+3 = 5$. e. $B^{(1)}$ is not a basis because the 2nd vector is parallel to the first. $B^{(2)}$ is a basis because $dim(W) = 2$ and these two vectors are not parallel to each other and both vectors are members of the Spanning set. $B^{(3)}$ is a basis for W for the same reason.

13. a. $R = \begin{bmatrix} 1 & -\frac{2}{3} & 0 & 0 & -\frac{11}{18} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ e. $R' = \begin{bmatrix} 1 & 0 & 0 & \frac{9}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b. $\{\langle 2,3,0,0,0\rangle, \langle 11,0,9,-15,18\rangle\}$
c. $\{\langle 18,-12,0,0,-11\rangle, \langle 0,0,2,0,-1\rangle, \langle 0,0,0,6,5\rangle\}$; d. $dim(W) = 3$ and $dim(W^{\perp}) = 2$; $3+2 = 5$.
f. $B^{(1)}$ is a basis because the first 3 columns of R' are linearly independent. $B^{(2)}$ is not, because the 4th column is dependent on the first two. $B^{(3)}$ is independent. Suppose $\vec{v}_4 = c_1\vec{v}_1 + c_3\vec{v}_3$ where ***neither*** c_1 nor c_3 is zero (notice, $\vec{v}_4$ is not parallel to either $\vec{v}_1$ or $\vec{v}_3$, so a dependence equation must involve both vectors). But $\vec{v}_4 = \frac{9}{4}\vec{v}_1 + \frac{1}{4}\vec{v}_2$. Setting these two equal, we would get a dependence equation for $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$, which is impossible. Similarly, $B^{(4)}$ is independent.

14. a. $R = \begin{bmatrix} 1 & 0 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & \frac{5}{6} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ e. $R' = \begin{bmatrix} 1 & 0 & 7 & 0 & 9 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ b. $\{\langle -10, -5, -4, 6\rangle\}$

c. $\{\langle 3,0,0,5\rangle, \langle 0,6,0,5\rangle, \langle 0,0,3,2\rangle\}$ d. $dim(W) = 3$ and $dim(W^{\perp}) = 1$; $3 + 1 = 4$.

f. $B^{(1)}$ is dependent, because the first three columns of R' are dependent. $B^{(2)}$ is independent, because $\vec{c}_1$, $\vec{c}_2$ and $\vec{c}_4$ of R' are independent. $B^{(3)}$ is independent. Suppose $\vec{v}_4 = c_2\vec{v}_2 + c_3\vec{v}_3$, again, where neither c_2 nor c_3 is zero. But we know that $\vec{v}_3 = 7\vec{v}_1 - 5\vec{v}_2$. Plugging this into the previous equation gives us a dependence equation for $\vec{v}_4$ with $\vec{v}_1$ and $\vec{v}_2$, which is impossible. $B^{(4)}$ is independent but the reasoning is a bit more complicated. Suppose $\vec{v}_5 = c_3\vec{v}_3 + c_4\vec{v}_4$. Replace $\vec{v}_3$ with $7\vec{v}_1 - 5\vec{v}_2$ as before, and distribute this over c_3. Replace $\vec{v}_5$ with $9\vec{v}_1 - 3\vec{v}_2 + 2\vec{v}_4$. Use the Uniqueness of Representation Property to get a contradiction.

15. a. $\{\langle 1, 0, 5, 0, 4\rangle, \langle 0, 1, -4, 0, 3\rangle, \langle 0, 0, 0, 1, -6\rangle\}$ b. $\{\langle 4, 3, 8, 5, -5\rangle, \langle 5, 7, -3, 6, 5\rangle, \langle 3, 4, -1, 4, 0\rangle\}$
c. $\{\langle -5, 4, 1, 0, 0\rangle, \langle -4, -3, 0, 6, 1\rangle\}$ d. $B^{(1)}$ is not a basis; $B^{(2)}$ is a basis; $B^{(3)}$ is a basis.

16. a. $\{\langle 3, 0, 0, -10, -2\rangle, \langle 0, 3, 0, 4, -1\rangle, \langle 0, 0, 3, -13, -2\rangle\}$
b. $\{\langle 5, 8, -3, 7, -4\rangle, \langle 3, 4, -2, 4, -2\rangle, \langle -8, -9, 5, -7, 5\rangle\}$
c. $\{\langle 10, -4, 13, 3, 0\rangle, \langle 2, 1, 2, 0, 3\rangle\}$ d. $B^{(1)}$ is not a basis; $B^{(2)}$ is a basis; $B^{(3)}$ is a basis.

17. a. $\{\langle 44, 0, 0, 0, 81\rangle, \langle 0, 66, 0, 0, 25\rangle, \langle 0, 0, 44, 0, -29\rangle, \langle 0, 0, 0, 132, -487\rangle\}$
b. $\{\langle 7, 4, -3, 2, 9\rangle, \langle 3, -5, 2, -1, 6\rangle, \langle 6, 9, -7, 3, 8\rangle, \langle 4, -2, 5, -1, 7\rangle\}$
c. $\{\langle -243, -50, 87, 487, 132\rangle\}$ d. $B^{(1)}$ is a basis; $B^{(2)}$ is a basis; $B^{(3)}$ is not a basis.

18. a. $\{\langle 1, 0, 7, -4, 0, 2\rangle, \langle 0, 1, -5, 3, 0, -5\rangle, \langle 0, 0, 0, 0, 1, 8\rangle\}$
b. $\{\langle 3, 5, -4, 3, 3, 5\rangle, \langle -2, -4, 6, -4, -2, 0\rangle, \langle 1, -1, 12, -7, -1, -1\rangle\}$
c. $\{\langle -7, 5, 1, 0, 0, 0\rangle, \langle 4, -3, 0, 1, 0, 0\rangle, \langle -2, 5, 0, 0, -8, 1\rangle\}$
d. $B^{(1)}$ is a basis; $B^{(2)}$ is a basis; $B^{(3)}$ is not a basis; $B^{(4)}$ is a basis.
The rref of $\begin{bmatrix} 1 & -1/2 & 2 \\ -1/2 & -1/2 & 1 \\ 1 & 1/2 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so the last 3 columns are linearly independent. Thus, $B^{(4)}$ is linearly independent.

19. a. No. b. Yes. c. Yes. d. No. e. No.
20. a. Yes. b. Yes. c. Yes. d. Yes. e. No.
21. a. Yes. b. Yes. c. Yes. d. No. e. Yes.
22. 22. a. Yes. b. Yes. c. No. d. Yes. e. Yes.
23. a. No. b. No. c. Yes. d. Yes. e. No.
24. a. No. b. Yes. c. No. d. Yes. e. No.
25. a. Yes. b. No. c. No. d. Yes. e. No.
26. a. Yes. b. Yes. c. Yes. d. Yes. e. No.
27. a. No. b. No. c. Yes. d. Yes. e. Yes.
36. a. True. b. True. c. False. d. True. e. False. f. True. g. True. h. True. i. False. j. True. k. False. l. True. m. False. n. False. o. True. p. True. q. False. r. False. s. True. t. True. u. True.

37. a. False. b. True. c. False. d. False. e. True. f. False. g. False. h. True. i. False. j. True. k. False. l. True. m. False. n. True. o. False. p. False. q. False. r. True. s. True. t. False.

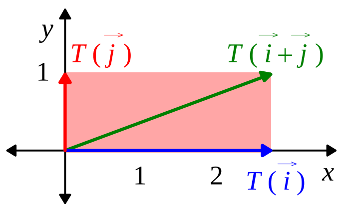
Chapter Two Exercises

2.1 Exercises

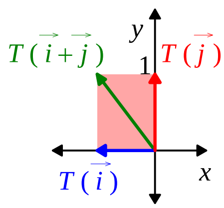
1. a. f is a function since every parent has a unique oldest child. b. g is not a function because x may not have any daughter at all. c. h is a function because every person has a unique mother. d. k is not a function because y may not have any brother at all. e. p is not a function because even though x has at least one child, none of the children of x may have any children of their own. f. q is a function because the father of y is unique, say call him z, and the mother of z is also unique.
2. a. $\langle -15, 38, 5\rangle$. c. $[T] = \begin{bmatrix} 2 & 3 \\ 1 & -5 \\ 4 & 1 \end{bmatrix}$.
3. a. $\langle -25, -6, -9\rangle$. c. $[T] = \begin{bmatrix} 2 & 0 & -5 & 0 \\ 0 & 3 & 1 & -2 \\ 3 & 8 & 0 & 0 \end{bmatrix}$.
4. a. $\langle 55, -21, 58, 84\rangle$. c. $[T] = \begin{bmatrix} 3 & 2 & -5 \\ 1 & 0 & 4 \\ 0 & 2 & -7 \\ 4 & 9 & 0 \end{bmatrix}$
5. a. $\langle 23, 62, -10\rangle$. c. $[T] = \begin{bmatrix} 5 & -3 & -2 \\ 4 & -6 & 3 \\ 2 & 2 & 0 \end{bmatrix}$
6. No. T is neither additive nor homogeneous.
7. No. T is neither additive nor homogeneous.
8. a. $[T] = \begin{bmatrix} 0 & 2 \\ -5 & 4 \\ 3 & -7 \end{bmatrix}$. b. $T(\langle x, y\rangle) = \langle 2y, -5x + 4y, 3x - 7y\rangle$ c. $\langle -4, -43, 35\rangle$.
9. a. $[T] = \begin{bmatrix} -3 & 2 & 0 \\ 5 & 7 & 4 \end{bmatrix}$. b. $T(\langle x, y, z\rangle) = \langle -3x + 2y, 5x + 7y + 4z\rangle$ c. $\langle -19, 35\rangle$.
10. a. $[T] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$.

b. $T(\langle x_1, x_2, x_3, x_4, x_5\rangle) = \langle x_5, x_3, x_1, x_4, x_2\rangle$ c. $\langle 9, -5, 3, 2, 0\rangle$.

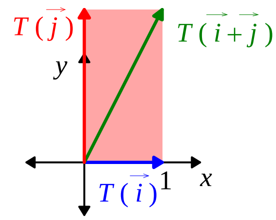
11. $T(\vec{v}_1) = \langle 6, -4, 17\rangle$ and $T(\vec{v}_2) = \langle -13, 10, -44\rangle$.



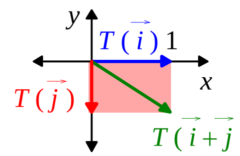
Exercise 12



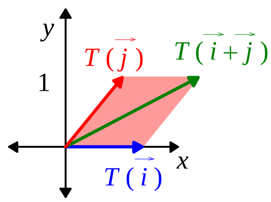
Exercise 13



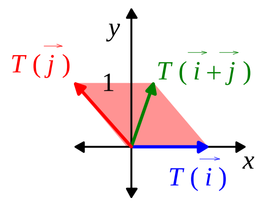
Exercise 14



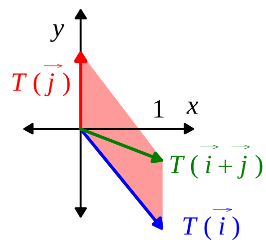
Exercise 15



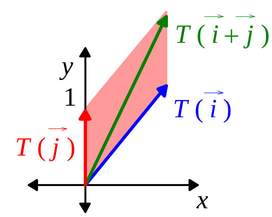
Exercise 16



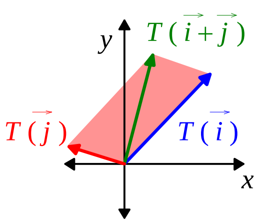
Exercise 17



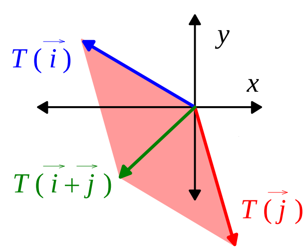
Exercise 18



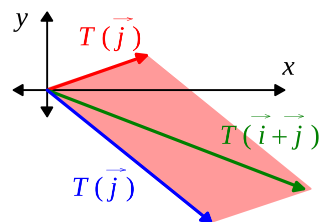
Exercise 19



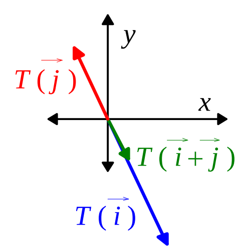
Exercise 20



Exercise 21

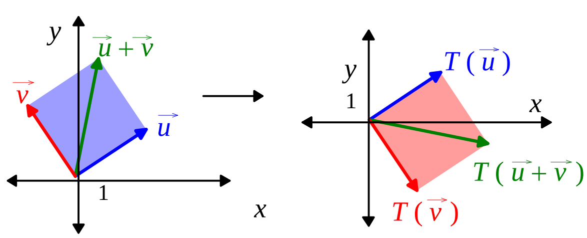


Exercise 22

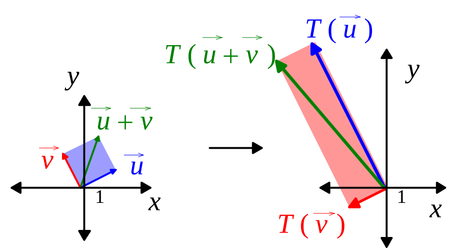


Exercise 23

23. The box "collapsed" into a line, because the two columns are parallel.



Exercise 24



Exercise 25

26. a. Yes. b. No. c. No. d. Yes. e. No. f. No. g. Yes. h. No. i. No. j. No. k. No. l. Yes.

29. $$[S_k] = \begin{bmatrix} k & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k \end{bmatrix}$$

2.2 Exercises

1. $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \left\langle \dfrac{5\sqrt{3}-3}{2}, \dfrac{3\sqrt{3}+5}{2} \right\rangle$

2. $\begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \langle 11/5, 27/5\rangle$

3. $\begin{bmatrix} -5/13 & -12/13 \\ 12/13 & -5/13 \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \langle -61/13, 45/13\rangle$

4. $\begin{bmatrix} 12/13 & -5/13 \\ 5/13 & 12/13 \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \langle 45/13, 61/13\rangle$

5. $\begin{bmatrix} -\frac{1}{2}\sqrt{2-\sqrt{2}} & -\frac{1}{2}\sqrt{\sqrt{2}+2} \\ \frac{1}{2}\sqrt{\sqrt{2}+2} & -\frac{1}{2}\sqrt{2-\sqrt{2}} \end{bmatrix}$;

$$rot_\theta(\langle 5,3\rangle) = \left\langle -\tfrac{3}{2}\sqrt{\sqrt{2}+2} - \tfrac{5}{2}\sqrt{-\sqrt{2}+2}, \tfrac{5}{2}\sqrt{\sqrt{2}+2} - \tfrac{3}{2}\sqrt{-\sqrt{2}+2} \right\rangle$$
$$\approx \langle -4.685, 3.471\rangle$$

6. $\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \left\langle \left(-5+3\sqrt{3}\right)/2, \left(-3-5\sqrt{3}\right)/2 \right\rangle$

7. $\begin{bmatrix} 21/29 & 20/29 \\ -20/29 & 21/29 \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \langle 165/29, -37/29\rangle$

8. $\begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \langle 27/5, -11/5\rangle$

9. $\begin{bmatrix} -8/17 & 15/17 \\ -15/17 & -8/17 \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \langle 5/17, -99/17\rangle$

10. $\begin{bmatrix} -\frac{41}{841} & \frac{840}{841} \\ -\frac{840}{841} & -\frac{41}{841} \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \left\langle \frac{2315}{841}, -\frac{4323}{841} \right\rangle \approx \langle 2.75, -5.14\rangle$

11. $[proj_L] = \begin{bmatrix} 25/34 & 15/34 \\ 15/34 & 9/34 \end{bmatrix}$; $[proj_{L^\perp}] = \begin{bmatrix} 9/34 & -15/34 \\ -15/34 & 25/34 \end{bmatrix}$;

$[refl_L] = \begin{bmatrix} 8/17 & 15/17 \\ 15/17 & -8/17 \end{bmatrix}$;

$proj_L(\langle 3,2\rangle) = \langle 105/34, 63/34\rangle$;

$proj_{L^\perp}(\langle 3,2\rangle) = \langle -3/34, 5/34\rangle$; $refl_L(\langle 3,2\rangle) = \langle 54/17, 29/17\rangle$

12. $[proj_L] = \begin{bmatrix} 49/65 & 28/65 \\ 28/65 & 16/65 \end{bmatrix}$; $[proj_{L^\perp}] = \begin{bmatrix} 16/65 & -28/65 \\ -28/65 & 49/65 \end{bmatrix}$;

$[refl_L] = \begin{bmatrix} 33/65 & 56/65 \\ 56/65 & -33/65 \end{bmatrix}$;

$proj_L(\langle 3,2\rangle) = \langle 203/65, 116/65\rangle$;
$proj_{L^\perp}(\langle 3,2\rangle) = \langle -8/65, 14/65\rangle$; $refl_L(\langle 3,2\rangle) = \langle 211/65, 102/65\rangle$

13. $[proj_L] = \begin{bmatrix} 25/41 & -20/41 \\ -20/41 & 16/41 \end{bmatrix}$; $[proj_{L^\perp}] = \begin{bmatrix} 16/41 & 20/41 \\ 20/41 & 25/41 \end{bmatrix}$;

$[refl_L] = \begin{bmatrix} 9/41 & -40/41 \\ -40/41 & -9/41 \end{bmatrix}$;

$proj_L(\langle 3,2\rangle) = \langle 35/41, -28/41\rangle$;
$proj_{L^\perp}(\langle 3,2\rangle) = \langle 88/41, 110/41\rangle$; $refl_L(\langle 3,2\rangle) = \langle -53/41, -138/41\rangle$

14. $[proj_L] = \begin{bmatrix} 9/58 & -21/58 \\ -21/58 & 49/58 \end{bmatrix}$; $[proj_{L^\perp}] = \begin{bmatrix} 49/58 & 21/58 \\ 21/58 & 9/58 \end{bmatrix}$;

$[refl_L] = \begin{bmatrix} -20/29 & -21/29 \\ -21/29 & 20/29 \end{bmatrix}$;

$proj_L(\langle 3,2\rangle) = \langle -15/58, 35/58\rangle$;
$proj_{L^\perp}(\langle 3,2\rangle) = \langle 189/58, 81/58\rangle$; $refl_L(\langle 3,2\rangle) = \langle -102/29, -23/29\rangle$

15. $[proj_L] = \begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{bmatrix} \begin{matrix} 3 \\ 2 \end{matrix}$; $[proj_{L^\perp}] = \begin{bmatrix} \frac{9}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{10} \end{bmatrix}$; $[refl_L] = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$;

$proj_L(\langle 3,2\rangle) = \langle 9/10, 27/10\rangle$;
$proj_{L^\perp}(\langle 3,2\rangle) = \langle 21/10, -7/10\rangle$; $refl_L(\langle 3,2\rangle) = \langle -6/5, 17/5\rangle$

16. $[proj_\Pi] = \frac{1}{29}\begin{bmatrix} 13 & -8 & 12 \\ -8 & 25 & 6 \\ 12 & 6 & 20 \end{bmatrix}$; $[proj_L] = \frac{1}{29}\begin{bmatrix} 16 & 8 & -12 \\ 8 & 4 & -6 \\ -12 & -6 & 9 \end{bmatrix}$;

$[refl_\Pi] = \frac{1}{29}\begin{bmatrix} -3 & -16 & 24 \\ -16 & 21 & 12 \\ 24 & 12 & 11 \end{bmatrix}$

17. $[proj_\Pi] = \frac{1}{65}\begin{bmatrix} 61 & 10 & -12 \\ 10 & 40 & 30 \\ -12 & 30 & 29 \end{bmatrix}$; $[proj_L] = \frac{1}{65}\begin{bmatrix} 4 & -10 & 12 \\ -10 & 25 & -30 \\ 12 & -30 & 36 \end{bmatrix}$;

$$[refl_\Pi] = \frac{1}{65}\begin{bmatrix} 57 & 20 & -24 \\ 20 & 15 & 60 \\ -24 & 60 & -7 \end{bmatrix}$$

18. $[proj_\Pi] = \frac{1}{90}\begin{bmatrix} 41 & 28 & 35 \\ 28 & 74 & -20 \\ 35 & -20 & 65 \end{bmatrix}$; $[proj_L] = \frac{1}{90}\begin{bmatrix} 49 & -28 & -35 \\ -28 & 16 & 20 \\ -35 & 20 & 25 \end{bmatrix}$;

$$[refl_\Pi] = \frac{1}{45}\begin{bmatrix} -4 & 28 & 35 \\ 28 & 29 & -20 \\ 35 & -20 & 20 \end{bmatrix}$$

19. $[proj_\Pi] = \frac{1}{34}\begin{bmatrix} 25 & 0 & -15 \\ 0 & 34 & 0 \\ -15 & 0 & 9 \end{bmatrix}$; $[proj_L] = \frac{1}{34}\begin{bmatrix} 9 & 0 & 15 \\ 0 & 0 & 0 \\ 15 & 0 & 25 \end{bmatrix}$;

$$[refl_\Pi] = \frac{1}{17}\begin{bmatrix} 8 & 0 & -15 \\ 0 & 17 & 0 \\ -15 & 0 & -8 \end{bmatrix}$$

20. $[proj_\Pi] = \frac{1}{53}\begin{bmatrix} 53 & 0 & 0 \\ 0 & 49 & 14 \\ 0 & 14 & 4 \end{bmatrix}$; $[proj_L] = \frac{1}{53}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & -14 \\ 0 & -14 & 49 \end{bmatrix}$;

$$[refl_\Pi] = \frac{1}{53}\begin{bmatrix} 53 & 0 & 0 \\ 0 & 45 & 28 \\ 0 & 28 & -45 \end{bmatrix}$$

21. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; No, because of the −1.

22. $[refl_L] = \frac{1}{19}\begin{bmatrix} -10 & -15 & 6 \\ -15 & 6 & -10 \\ 6 & -10 & -15 \end{bmatrix} = -[refl_\Pi]$.

23. $[refl_L] = \frac{1}{65}\begin{bmatrix} -57 & -20 & 24 \\ -20 & -15 & -60 \\ 24 & -60 & 7 \end{bmatrix}$

24. a. $T(\vec{v}) = \langle 2, 5\rangle$ and $T(\vec{w}) = \langle 4, -3\rangle$. c. it corresponds to $refl_L$

e. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is the matrix of the reflection across $y = z$, and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is the matrix of the reflection across $x = z$. f. $T(\langle x_1, x_2, x_3, x_4\rangle) = \langle x_1, x_4, x_3, x_2\rangle$; T exchanges the 2nd and 4th components of $\vec{v}$.

27. $6x - 3y + 8z = 0$.

28. a. $\sqrt{29}/\sqrt{38}$, $\sqrt{13}/\sqrt{38}$, $\sqrt{34}/\sqrt{38}$. The radicand in the numerator is the respective diagonal entry.

b. $\frac{15}{38}$; $\frac{-6}{38}$; $\frac{10}{38}$; c. $\cos(\alpha_{i,j}) = \frac{15}{\sqrt{377}}$; $\alpha_{i,j} = \cos^{-1}\left(\frac{15}{\sqrt{377}}\right) \approx 39.42^0$

$\cos(\alpha_{i,k}) = \frac{-6}{\sqrt{986}}$; $\alpha_{i,k} = \cos^{-1}\left(\frac{-6}{\sqrt{986}}\right) \approx 101.02^0$; $\cos(\alpha_{j,k}) = \frac{10}{\sqrt{442}}$;

$\alpha_{j,k} = \cos^{-1}\left(\frac{10}{\sqrt{442}}\right) \approx 61.60^0$

2.3 Exercises

1. a. $(T_1 + T_2)(\langle x, y, z\rangle) = \langle 5x - 2y + 14z, 2x + 3y - 4z\rangle$. b. $[T_1 + T_2] = \begin{bmatrix} 5 & -2 & 14 \\ 2 & 3 & -4 \end{bmatrix}$

c. $[T_1] = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 4 & -7 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 2 & 0 & 9 \\ 1 & -1 & 3 \end{bmatrix}$

e. $[-4T_1] = \begin{bmatrix} -12 & 8 & -20 \\ -4 & -16 & 28 \end{bmatrix} = -4[T_1]$.

2. a. $(T_1 + T_2)(\langle x, y, z\rangle) = \langle 3x - 2y + 4z, 2x - y - 4z, x + 2y + 3z, -3x - y + z\rangle$.

b. $\begin{bmatrix} 3 & -2 & 4 \\ 2 & -1 & -4 \\ 1 & 2 & 3 \\ -3 & -1 & 1 \end{bmatrix}$ c. $[T_1] = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & -4 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 3 \\ -4 & 0 & 0 \end{bmatrix}$ e. $\begin{bmatrix} 3 & -6 & 9 \\ 3 & 0 & -12 \\ 0 & 6 & 0 \\ 3 & -3 & 3 \end{bmatrix}$

3. The matrices that do not exist are: b. $A - B$ d. $7C + 4A$ f. CB h. BE. The matrices that exist, and their sizes, are:

a. $\begin{bmatrix} -2 & -4 & -3 \\ 6 & 7 & -5 \end{bmatrix}$ 2×3

c. $\begin{bmatrix} -3 & -11 \\ 4 & 26 \\ -29 & -15 \end{bmatrix}$ 3×2

e. $\begin{bmatrix} -32 & 22 \\ 43 & -19 \\ -4 & -7 \end{bmatrix}$ 3×2

g. $\begin{bmatrix} 57 & -40 \\ -20 & 17 \end{bmatrix}$ 2×2

i. $\begin{bmatrix} 3 & 31 & -13 \\ -2 & -46 & 20 \\ 17 & -27 & 17 \end{bmatrix}$ 3×3

j. $\begin{bmatrix} 55 & 34 \\ -19 & 15 \end{bmatrix}$ 2×2

k. $\begin{bmatrix} 317 & -118 \\ -163 & 121 \end{bmatrix}$ 2×2

m. $\begin{bmatrix} -13 & 195 & -91 \\ 26 & -314 & 148 \\ 65 & -367 & 183 \end{bmatrix}$ 3×3

o. $\begin{bmatrix} 461 & 178 \\ -167 & -23 \end{bmatrix}$ 2×2

4. a. $\begin{bmatrix} 1 & 8 & -15 \\ 37 & -52 & -69 \\ -28 & -17 & 2 \end{bmatrix}$; 3×3

b. $\begin{bmatrix} 56 & 5 & -35 & 55 \\ -1 & -29 & 18 & 16 \\ -3 & -24 & -13 & 39 \\ -39 & 4 & 41 & -63 \end{bmatrix}$; 4×4

c. $\begin{bmatrix} 5 & -15 & -15 & 13 & 70 \\ 93 & -35 & 63 & -49 & 88 \\ -63 & -15 & -14 & 31 & -4 \end{bmatrix}$; 3×5

d. does not exist

e. $\begin{bmatrix} 13 & -56 & 72 \\ 52 & -31 & -41 \\ -63 & 50 & 10 \\ 37 & -29 & -60 \end{bmatrix}$; 4×3

f. $\begin{bmatrix} -50 & -53 & 65 & -25 \\ 23 & 1 & 0 & 10 \\ 64 & 26 & -20 & 20 \\ -11 & -17 & -12 & 26 \\ -16 & -6 & 20 & -20 \end{bmatrix}$; 5×4

g. $\begin{bmatrix} 41 & -51 & -84 \\ -19 & 20 & 41 \\ 14 & -17 & -60 \\ 12 & 2 & 31 \\ 41 & 36 & -9 \end{bmatrix}$; 5×3

h. does not exist

i. $\begin{bmatrix} 89 & 59 & -59 & 30 \\ -17 & -49 & -21 & 0 \\ -85 & 27 & 139 & -58 \\ 71 & 6 & -75 & -4 \end{bmatrix}$; 4×4

j. does not exist.

k. $\begin{bmatrix} 631 & -225 & 362 & -299 & 672 \\ -237 & -105 & -101 & 163 & 194 \\ 14 & -250 & 247 & -54 & 272 \\ -550 & 310 & -477 & 312 & -622 \end{bmatrix}$; 4×5

l. same as k

m. $\begin{bmatrix} 503 & -1 & -356 \\ -139 & -207 & -326 \\ -425 & 649 & 1340 \\ 306 & 83 & -560 \end{bmatrix}$; 4×3 (same as part n)

o. $\begin{bmatrix} 717 & -153 & -597 \\ 45 & 4173 & 2895 \\ -713 & 626 & 1597 \end{bmatrix}$; 3×3

5. a. The codomain of T_1 is $\mathbb{R}^4$, which is also the domain of T_2. The domain of $T_2 \circ T_1$ is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^3$

   b. This is not well defined. c. $\langle 9x - 26y, 33x + 9y, -6x + 54y \rangle$

   d. $\begin{bmatrix} 9 & -26 \\ 33 & 9 \\ -6 & 54 \end{bmatrix}$ e. $[T_2] = \begin{bmatrix} 3 & 0 & 0 & -5 \\ 0 & 7 & 2 & -1 \\ 0 & 0 & 6 & 9 \end{bmatrix}$; $[T_1] = \begin{bmatrix} 3 & -2 \\ 5 & 1 \\ -1 & 3 \\ 0 & 4 \end{bmatrix}$;

   $[T_2][T_1] = \begin{bmatrix} 9 & -26 \\ 33 & 9 \\ -6 & 54 \end{bmatrix}$

6. a. The codomain of one is the domain of the other, so both compositions are well-defined. $T_2 \circ T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and $T_1 \circ T_2 : \mathbb{R}^4 \to \mathbb{R}^4$.

   b. $(T_2 \circ T_1)(\langle x, y, z \rangle) = \langle 9x + 10y + 7z, 16x - 8y + 32z, 6x + 9y - 12z \rangle$, and $T_1 \circ T_2(\langle x_1, x_2, x_3, x_4 \rangle) = \langle 9x_1 + 35x_2 + 4x_3 - 29x_4, 6x_1 - 7x_2 + 22x_3 + 27x_4, 3x_1 + 6x_3 + 4x$

   c. $[T_2 \circ T_1] = \begin{bmatrix} 9 & 10 & 7 \\ 16 & -8 & 32 \\ 6 & 9 & -12 \end{bmatrix}$, $[T_1 \circ T_2] = \begin{bmatrix} 9 & 35 & 4 & -29 \\ 6 & -7 & 22 & 27 \\ 3 & 0 & 6 & 4 \\ 0 & 7 & -10 & -19 \end{bmatrix}$

   d. $[T_2] = \begin{bmatrix} 3 & 0 & 0 & -5 \\ 0 & 7 & 2 & -1 \\ 0 & 0 & 6 & 9 \end{bmatrix}$; $[T_1] = \begin{bmatrix} 3 & 5 & -1 \\ 2 & -1 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$;

   $[T_2][T_1] = \begin{bmatrix} 9 & 10 & 7 \\ 16 & -8 & 32 \\ 6 & 9 & -12 \end{bmatrix}$; $[T_1][T_2] = \begin{bmatrix} 9 & 35 & 4 & -29 \\ 6 & -7 & 22 & 27 \\ 3 & 0 & 6 & 4 \\ 0 & 7 & -10 & -19 \end{bmatrix}$

7. a. The codomain of one is the domain of the other, so both compositions are well-defined. $T_2 \circ T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_1 \circ T_2 : \mathbb{R}^5 \to \mathbb{R}^5$. b. $(T_2 \circ T_1)(\langle x, y \rangle) = \langle 10x - 13y, 17x + 26y \rangle$, and $(T_1 \circ T_2)(\langle x_1, x_2, x_3, x_4, x_5 \rangle) = \langle 21x_1 + 7x_2 - 2x_3 + 3x_4 - 6x_5, 21x_2 - 20x_3 + 16x_4 - 25x_5,$ $78x_1 + 35x_2 - 16x_3 + 18x_4 - 33x_5, 54x_1 + 12x_3 - 6x_4 + 6x_5, -6x_1 - 14x_2 + 12x_3 - 10x_4 + 16x$

c. $[T_2 \circ T_1] = \begin{bmatrix} 10 & -13 \\ 17 & 26 \end{bmatrix}$; $[T_1 \circ T_2] = \begin{bmatrix} 21 & 7 & -2 & 3 & -6 \\ 0 & 21 & -20 & 16 & -25 \\ 78 & 35 & -16 & 18 & -33 \\ 54 & 0 & 12 & -6 & 6 \\ -6 & -14 & 12 & -10 & 16 \end{bmatrix}$

d. $[T_2] = \begin{bmatrix} 3 & 7 & -6 & 5 & -8 \\ 9 & 0 & 2 & -1 & 1 \end{bmatrix}$

$[T_1] = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 5 & 7 \\ 0 & 6 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 7 & -6 & 5 & -8 \\ 9 & 0 & 2 & -1 & 1 \end{bmatrix}$; $[T_2][T_1] = \begin{bmatrix} 10 & -13 \\ 17 & 26 \end{bmatrix}$;

$[T_1][T_2] = \begin{bmatrix} 21 & 7 & -2 & 3 & -6 \\ 0 & 21 & -20 & 16 & -25 \\ 78 & 35 & -16 & 18 & -33 \\ 54 & 0 & 12 & -6 & 6 \\ -6 & -14 & 12 & -10 & 16 \end{bmatrix}$

13. If A is $m \times k$, then B has to be $k \times m$. For both compositions to be defined, m must equal n.

2.4 Exercises

1. a. $\begin{bmatrix} 11 & -7 & -1 & 11 \\ -6 & 4 & 0 & 1 \\ -7 & 13 & 8 & 4 \end{bmatrix}$ b. $\begin{bmatrix} 96 & 138 \\ -54 & -32 \\ -72 & 5 \end{bmatrix}$ c. $\begin{bmatrix} 56 & 75 \\ 0 & 77 \\ -40 & -15 \end{bmatrix}$

d. $\begin{bmatrix} 40 & 63 \\ -54 & -109 \\ -32 & 20 \end{bmatrix}$ e. $\begin{bmatrix} 96 & 138 \\ -54 & -32 \\ -72 & 5 \end{bmatrix}$

f. $\begin{bmatrix} 2 & 12 \\ -3 & -2 \\ 0 & 6 \\ 8 & 7 \end{bmatrix}$ g. $\begin{bmatrix} 48 & 59 \\ -64 & -132 \\ 5 & 8 \end{bmatrix}$ h. $\begin{bmatrix} 8 & -4 \\ -10 & -23 \\ 37 & -12 \end{bmatrix}$

i. $\begin{bmatrix} 48 & 59 \\ -64 & -132 \\ 5 & 8 \end{bmatrix}$ j. $\begin{bmatrix} 131 & 217 \\ -16 & -73 \\ -21 & -34 \end{bmatrix}$

2. a. $[T_1] = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 5 & -7 \\ 1 & -1 & 4 \\ 6 & 1 & -1 \end{bmatrix}$; 4×3; $[T_2] = \begin{bmatrix} 5 & 0 & 2 & -1 \\ 2 & 8 & -6 & 7 \end{bmatrix}$; 2×4;

$[T_3] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 7 & 3 \\ 4 & 1 \\ 1 & 5 \end{bmatrix}$; 5×2

b. $(T_2 \circ T_1)(\langle x, y, z \rangle) = \langle 6x - 18y + 9z, 40x + 47y - 87z \rangle$.

c. $\begin{bmatrix} 6 & -18 & 9 \\ 40 & 47 & -87 \end{bmatrix}$; 2×3; d. same as c.

e. $[T_3 \circ T_2] = \begin{bmatrix} 9 & 16 & -10 & 13 \\ 3 & -8 & 8 & -8 \\ 41 & 24 & -4 & 14 \\ 22 & 8 & 2 & 3 \\ 15 & 40 & -28 & 34 \end{bmatrix}$ (5×4);

f. $[T_3 \circ T_2 \circ T_1] = \begin{bmatrix} 86 & 76 & -165 \\ -34 & -65 & 96 \\ 162 & 15 & -198 \\ 64 & -25 & -51 \\ 206 & 217 & -426 \end{bmatrix}$ (5×3).

3. a. $[T_1] = \begin{bmatrix} 8/17 & 15/17 \\ 15/17 & -8/17 \end{bmatrix}$; $[T_2] = \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$;

$[T_3] = \begin{bmatrix} 9/58 & -21/58 \\ -21/58 & 49/58 \end{bmatrix}.$

b. $[T_2 \circ T_1] = \begin{bmatrix} \frac{84}{85} & \frac{13}{85} \\ \frac{13}{85} & -\frac{84}{85} \end{bmatrix}$; $[T_1 \circ T_3] = \begin{bmatrix} -\frac{243}{986} & \frac{567}{986} \\ \frac{303}{986} & -\frac{707}{986} \end{bmatrix}$

c. $[T_3 \circ T_2 \circ T_1] = \begin{bmatrix} \frac{483}{4930} & \frac{1881}{4930} \\ -\frac{1127}{4930} & -\frac{4389}{4930} \end{bmatrix}$; $[T_1 \circ T_3 \circ T_2] = \begin{bmatrix} -\frac{2997}{4930} & \frac{729}{4930} \\ \frac{3737}{4930} & -\frac{909}{4930} \end{bmatrix}$;

we get different answers.

4. $[T_1] = \begin{bmatrix} 2 & -3 & 1 \\ 4 & -5 & -7 \end{bmatrix} (2 \times 3)$, $[T_2] = \begin{bmatrix} 5 & -4 \\ 1 & -3 \\ 7 & 2 \end{bmatrix} (3 \times 2)$,

$[T_1 \circ T_2] = \begin{bmatrix} 14 & 3 \\ -34 & -15 \end{bmatrix} (2 \times 2)$, $[T_2 \circ T_1] = \begin{bmatrix} -6 & 5 & 33 \\ -10 & 12 & 22 \\ 22 & -31 & -7 \end{bmatrix} (3 \times 3)$.

5. $A^2 = \begin{bmatrix} 44 & -35 \\ -25 & 39 \end{bmatrix}$, $A^3 = \begin{bmatrix} -307 & 378 \\ 270 & -253 \end{bmatrix}$, $A^4 = \begin{bmatrix} 2811 & -2905 \\ -2075 & 2396 \end{bmatrix}$.

$p(A) = 4I_2 - 6A + 5A^2 - 2A^3 + 7A^4 = \begin{bmatrix} 20,533 & -21,308 \\ -15,220 & 17,489 \end{bmatrix}$. Reminder: the first term is $4I_2$.

6. $A^2 = \begin{bmatrix} 3 & -8 & -16 \\ 0 & 1 & -6 \\ 4 & 24 & 51 \end{bmatrix}$, $A^3 = \begin{bmatrix} -5 & -56 & -118 \\ 9 & -25 & -42 \\ 25 & 180 & 349 \end{bmatrix}$, $p(A) = \begin{bmatrix} -15 & -72 & -170 \\ 39 & -59 & -54 \\ 23 & 268 & 495 \end{bmatrix}$

7. a. We have two non-zero, non-parallel vectors. b. $\langle 217, 579, -694 \rangle$

8. a. The rref of the matrix with the 3 vectors as columns is I_3. b. $\langle 18, \frac{19}{2}, \frac{57}{2}, -7, \frac{59}{2} \rangle$

16. a. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$; rotate $\mathbb{R}^2$ by θ, then reflect $\mathbb{R}^2$ across the y-axis.

b. $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; reflect $\mathbb{R}^2$ across the x-axis, then rotate $\mathbb{R}^2$ by θ.

19. Rotating $\mathbb{R}^2$ by α, followed by another rotation by β results in a net rotation by $\alpha + \beta$. Similarly, rotating $\mathbb{R}^2$ by β, followed by another rotation by α results in a net rotation by $\beta + \alpha$, which is the same as $\alpha + \beta$.

2.5 Exercises

1. a. $[T_1] = \begin{bmatrix} 3 & 1 & -7 & 8 \\ 2 & 2 & -2 & -4 \\ -2 & 1 & 8 & -17 \end{bmatrix}$ b. $R_1 = \begin{bmatrix} 1 & 0 & -3 & 5 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
   c. $\{\langle 3,-2,1,0\rangle, \langle -5,7,0,1\rangle\}$ d. $nullity(T_1) = 2$
   e. T_1 is not 1-1. f. $\{\langle 3,2,-2\rangle, \langle 1,2,1\rangle\}$ g. $rank(T_1) = 2$ h. T_1 is not onto. i. $2+2 = 4$.
2. a. $[T_2] = \begin{bmatrix} 3 & -6 & 5 \\ 2 & -4 & 7 \\ -5 & 10 & 3 \\ -1 & 2 & 8 \end{bmatrix}$ b. $R_2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ c. $\{\langle 2,1,0\rangle\}$
   d. $nullity(T_2) = 1$; e. T_2 is not 1-1.
   f. $\{\langle 3,2,-5,-1\rangle, \langle 5,7,3,8\rangle\}$ g. $rank(T_2) = 2$. h. T_2 is not onto. i. $2+1 = 3$.
3. a. $[T_3] = \begin{bmatrix} -5 & -7 & 2 \\ -2 & 1 & 16 \\ 3 & -2 & -26 \end{bmatrix}$ b. $R_3 = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$
   c. $\{\langle 6,-4,1\rangle\}$ d. $nullity(T_3) = 1$; e. T_3 is not 1-1.
   f. $\{\langle -5,-2,3\rangle, \langle -7,1,-2\rangle\}$ g. $rank(T) = 2$. h. T_3 is not onto. i. $2+1 = 3$ j.
   The kernel is a line with direction $\langle 6,-4,1\rangle$, and the range is a plane with equation $x - 31y - 19z = 0$ k. The kernel is not necessarily orthogonal to the range (columnspace). The kernel is always orthogonal to the ***rowspace***.
4. a. $\{\langle -5,2,1\rangle\}$ b. 1 c. T is not one-to-one. d. $\{\langle 2,3,3,-3,3\rangle, \langle 3,4,5,2,10\rangle\}$
   e. 2. f. T is not onto. g. not full-rank. h. $2+1 = 3$.
5. a. there is no basis for the kernel of T. b. 0 c. T is one-to-one.
   d. $\{\langle 2,3,3,-3,3\rangle, \langle 3,4,5,2,10\rangle, \langle 4,7,5,-18,-5\rangle\}$
   e. 3 f. T is not onto. g. full-rank. h. $3+0 = 3$.
6. a. $\{\langle -4,-9,1,0,0\rangle, \langle 5,3,0,1,0\rangle, \langle -2,1,0,0,1\rangle\}$ b. 3 c. T is not one-to-one.
   d. $\{\langle 3,-5,-8\rangle, \langle -2,3,5\rangle\}$ e. 2 f. T is not onto. g. not full-rank. h. $2+3 = 5$.
7. a. $\{\langle -4,-9,1,0,0\rangle, \langle 5,3,0,1,0\rangle\}$ b. 2 c. T is not one-to-one.
   d. $\{\langle 3,-5,-8\rangle, \langle -2,3,5\rangle, \langle 8,-13,-20\rangle\}$ e. 3 f. T is onto. g. full-rank. h. $3+2 = 5$.
8. a. $\{\langle -2,-3,1,1,0\rangle, \langle 1,-2,5,0,1\rangle\}$ b. 2 c. T is not one-to-one.
   d. $\{\langle 3,-5,-8\rangle, \langle -2,3,5\rangle, \langle -2,9,4\rangle\}$ e. 3 f. T is onto. g. full-rank. h. $3+2 = 5$.
9. a. $\{\langle -4,-9,1,0,0\rangle, \langle -3,8,0,-5,1\rangle\}$ b. 2 c. T is not one-to-one.
   d. $\{\langle 3,-5,-8,6\rangle, \langle -2,3,5,-3\rangle, \langle -2,9,10,-8\rangle\}$
   e. 3 f. T is not onto. g. not full-rank. h. $3+2 = 5$.
10. a. $\{\langle 3,1,0,0,0\rangle, \langle 7,0,-5,1,0\rangle\}$ b. 2 c. T is not one-to-one.
    d. $\{\langle 3,-5,-2,2\rangle, \langle 6,-7,-3,5\rangle, \langle -2,9,7,-8\rangle\}$
    e. 3 f. T is not onto. g. not full-rank. h. $3+2 = 5$.
11. a. $\{\langle -2,1,-3,-5,1\rangle\}$ b. 1 c. T is not one-to-one.
    d. $\{\langle 3,-5,-2,2\rangle, \langle 6,-7,-3,5\rangle, \langle -2,3,4,7\rangle, \langle -1,-4,3,-2\rangle\}$
    e. 4 f. T is onto. g. full-rank. h. $4+1 = 5$.
12. a. $\{\langle -5,2,1,0,0\rangle\}$ b. 1 c. T is not one-to-one.

d. $\{\langle 3,-5,-2,2\rangle,\langle 6,-7,-3,5\rangle,\langle -2,3,4,7\rangle,\langle -1,-4,3,-2\rangle\}$
e. 4 f. T is onto. g. full-rank. h. $4+1=5$.

13. a. $\{\langle -72,25,45,0\rangle,\langle -36,35,0,45\rangle\}$ b. 2 c. T is not one-to-one.
d. $\{\langle 15,30,-10,-5,-15\rangle,\langle 72,63,27,-54,0\rangle\}$
e. 2 f. T is not onto. g. not full-rank. h. $2+2=4$.
14. a. there is no basis for the kernel of T b. 0 c. T is one-to-one.
d. $\{\langle 1,3,-1,-5,5\rangle,\langle 2,6,7,-4,0\rangle,\langle -6,3,-3,2,-4\rangle,\langle -4,-5,-2,3,1\rangle\}$
e. 4 f. T is not onto. g. full-rank. h. $4+0=4$.
15. a. $\{\langle -4,3,-2,1\rangle\}$ b. 1 c. T is not one-to-one.
d. $\{\langle 5,2,-6,-2,1\rangle,\langle 7,-1,-3,3,0\rangle,\langle 2,3,-5,1,-1\rangle\}$
e. 3 f. T is not onto. g. not full-rank. h. $3+1=4$.
16. a. $\{\langle 3,1,0,0\rangle,\langle -4,0,2,1\rangle\}$ b. 2 c. T is not one-to-one.
d. $\{\langle 2,3,2,5\rangle,\langle 3,1,5,4\rangle\}$ e. 2 f. T is not onto. g. not full-rank. h. $2+2=4$.
17. a. $\{\langle 5,-3,-8,1\rangle\}$ b. 1 c. T is not one-to-one.
d. $\{\langle 4,5,-6,5\rangle,\langle 2,9,-7,6\rangle,\langle 1,-2,-1,3\rangle\}$
e. 3 f. T is not onto. g. not full-rank. h. $3+1=4$.
18. a. $\{\langle 5,1,0,0,0\rangle,\langle -9,0,7,1,0\rangle\}$ b. 2 c. T is not one-to-one.
d. $\{\langle -3,2,5,0,-4\rangle,\langle -5,-1,2,-3,-7\rangle,\langle 12,-4,0,-25,37\rangle\}$
e. 3 f. T is not onto. g. not full-rank. h. $3+2=5$.
19. a. $\{\langle 7,-5,1,0,0\rangle\}$ b. 1 c. T is not one-to-one.
d. $\{\langle -3,2,4,0,-3\rangle,\langle -5,-1,6,-1,-4\rangle,\langle 2,-4,-5,-5,3\rangle,\langle -5,-1,2,-3,-7\rangle\}$
e. 4 f. T is not onto. g. not full-rank. h. $4+1=5$.
20. a. $\{\langle -7,2,-3,1,0\rangle,\langle 5,-3,2,0,1\rangle\}$ b. 2 c. T is not one-to-one.
d. $\{\langle -3,2,4,0,-3\rangle,\langle -5,-1,2,-3,-7\rangle,\langle 2,-4,-5,-5,3\rangle\}$
e. 3 f. T is not onto. g. not full-rank. h. $3+2=5$.
22. a. Π b. L c. L d. Π e. $\left\{\vec{\mathbf{0}}_3\right\}$ f. $\mathbb{R}^3$
31. The three image vectors are linearly dependent:
$\frac{8}{5}\langle 2,-3,4,-1,7\rangle-\frac{3}{5}\langle -3,2,-1,4,2\rangle=\langle 5,-6,7,-4,10\rangle$, so
$\frac{8}{5}\langle 1,-2,1\rangle-\frac{3}{5}\langle 0,-1,3\rangle-\langle 0,-2,5\rangle=\left\langle \frac{8}{5},-\frac{3}{5},-\frac{26}{5}\right\rangle$ is a non-zero vector in $ker(T)$.
32. a. True. b. False. c. True. d. False. e. False. f. True. g. True. h. True. i. False. j. True. k. False. l. False. m. True. n. True.

2.6 Exercises

1. $\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$ 2. $\begin{bmatrix} \frac{1}{5} & \frac{7}{20} \\ 0 & -\frac{1}{4} \end{bmatrix}$ 3. $\begin{bmatrix} 0 & \frac{1}{6} \\ \frac{1}{4} & 0 \end{bmatrix}$

4. $\begin{bmatrix} 4 & -9 \\ -3 & 7 \end{bmatrix}$ 5. $\begin{bmatrix} -1 & -2 \\ -\frac{4}{3} & -\frac{7}{3} \end{bmatrix}$ 6. $\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & -\frac{3}{8} \end{bmatrix}$

7. $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ 8. not invertible. 9. $\begin{bmatrix} \frac{11}{19} & -\frac{5}{57} \\ \frac{14}{19} & \frac{4}{57} \end{bmatrix}$

10. $\begin{bmatrix} -\frac{27}{124} & \frac{11}{124} \\ \frac{12}{31} & \frac{2}{31} \end{bmatrix}$ 11. $\begin{bmatrix} \frac{105}{179} & -\frac{24}{179} \\ \frac{10}{179} & \frac{100}{179} \end{bmatrix}$

12. $\frac{1}{24}\begin{bmatrix} -\sqrt{6} & \sqrt{30} \\ 2\sqrt{15} & -2\sqrt{3} \end{bmatrix}$ 13. $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ which is the matrix of the clockwise rotation by θ.

14. $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ 15. $\frac{1}{5}\begin{bmatrix} 3e^{-3x} & e^{2x} \\ -2e^{-4x} & e^{x} \end{bmatrix}$ 16. $\frac{1}{2 \cdot 60^x}\begin{bmatrix} 10^x & 4^x \\ 15^x & -6^x \end{bmatrix}$

17. $\begin{bmatrix} \cosh(x) & -\sinh(x) \\ -\sinh(x) & \cosh(x) \end{bmatrix}$ 18. not invertible. 19. $\begin{bmatrix} a^2 - b^2 & 2ab \\ 2ab & b^2 - a^2 \end{bmatrix}$

20. The projection operator (Exercise 18) is not invertible because the kernel consists of more than just the zero vector. The reflection operator (Exercise 19) is invertible because the kernel is only the zero vector. Furthermore, notice that the inverse is ***itself***, for the reason that the reflection of the reflection of a vector is the original vector.

21. $[T] = \begin{bmatrix} 3 & -7 \\ -4 & 9 \end{bmatrix}$; $[T]^{-1} = \begin{bmatrix} -9 & -7 \\ -4 & -3 \end{bmatrix}$; $T^{-1}(\langle x,y \rangle) = \langle -9x - 7y, -4x - 3y \rangle$

22. T is not invertible. 23. $[T] = \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}$; $[T]^{-1} = \begin{bmatrix} \frac{9}{2} & -\frac{5}{2} \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$;

$T^{-1}(\langle x,y \rangle) = \langle 9x/2 - 5y/2, -5x/2 + 3y/2 \rangle$.

24. $[T] = \begin{bmatrix} \frac{2}{3} & \frac{5}{3} \\ \frac{4}{3} & -\frac{1}{3} \end{bmatrix}$; $[T]^{-1} = \begin{bmatrix} \frac{3}{22} & \frac{15}{22} \\ \frac{6}{11} & -\frac{3}{11} \end{bmatrix}$;

$T^{-1}(\langle x,y \rangle) = \langle 3x/22 + 15y/22, 6x/11 - 3y/11 \rangle$.

26. a. $\begin{bmatrix} 6 & -21 \\ 10 & -35 \end{bmatrix}$; No. e. $\begin{bmatrix} 31 & -27 \\ -59 & 69 \end{bmatrix}$; Yes. f. $\begin{bmatrix} 31 & 124 \\ -93 & -372 \end{bmatrix}$; No. g. No.

b. You will never get an invertible matrix. 27. b. not invertible. It is not one-to-one.

2.7 Exercises

Note: answers vary for (b) and (c) in Exercises 1 to 12, so only answers to (a) are provided.

1. $\begin{bmatrix} 2 & -\frac{7}{3} \\ -1 & \frac{4}{3} \end{bmatrix}$ 2. $\begin{bmatrix} \frac{7}{11} & -\frac{3}{11} \\ \frac{10}{11} & \frac{2}{11} \end{bmatrix}$ 3. $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 1 \\ 1 & -1 & -1 \\ \frac{9}{5} & -\frac{12}{5} & -2 \end{bmatrix}$

4. not invertible. 5. $\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{11}{6} \\ 0 & -\frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

6. $\begin{bmatrix} 2 & 0 & 0 \\ \frac{3}{4} & \frac{3}{2} & 0 \\ \frac{13}{12} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix}$ 7. $\begin{bmatrix} \frac{5}{31} & -\frac{2}{31} & -\frac{4}{31} \\ -\frac{1}{62} & \frac{19}{62} & \frac{7}{62} \\ \frac{8}{31} & \frac{3}{31} & \frac{6}{31} \end{bmatrix}$

8. $\begin{bmatrix} \frac{8}{27} & \frac{2}{27} & \frac{11}{27} \\ \frac{7}{27} & -\frac{5}{27} & \frac{13}{27} \\ -\frac{4}{9} & -\frac{1}{9} & -\frac{1}{9} \end{bmatrix}$ 9. $\begin{bmatrix} \frac{3}{7} & \frac{1}{7} & \frac{4}{7} \\ 1 & -1 & 2 \\ -\frac{9}{7} & -\frac{3}{7} & \frac{2}{7} \end{bmatrix}$

10. $\begin{bmatrix} -1 & \frac{3}{2} & 11 & \frac{25}{6} \\ 0 & \frac{1}{2} & 2 & \frac{1}{6} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$ 11. not invertible.

12. $\begin{bmatrix} \frac{4}{7} & \frac{11}{14} & -\frac{13}{14} & -\frac{6}{7} \\ \frac{1}{7} & \frac{4}{7} & -\frac{6}{7} & -\frac{5}{7} \\ \frac{5}{7} & \frac{6}{7} & -\frac{9}{7} & -\frac{11}{7} \\ -\frac{2}{7} & -\frac{9}{14} & \frac{3}{14} & \frac{3}{7} \end{bmatrix}$ 14. $\begin{bmatrix} \frac{55}{31} \\ -\frac{73}{62} \\ \frac{26}{31} \end{bmatrix}$

15. $\begin{bmatrix} -\frac{20}{9} \\ -\frac{31}{9} \\ -\frac{2}{3} \end{bmatrix}$ 16. $\begin{bmatrix} \frac{4}{7} & -\frac{24}{7} \\ -4 & -18 \\ \frac{16}{7} & -\frac{26}{7} \end{bmatrix}$

17. $\begin{bmatrix} -\frac{253}{6} \\ -\frac{37}{6} \\ -\frac{11}{3} \\ -\frac{1}{3} \end{bmatrix}$ 18. $\begin{bmatrix} -\frac{62}{7} & \frac{179}{14} \\ -\frac{61}{7} & \frac{60}{7} \\ -\frac{102}{7} & \frac{132}{7} \\ \frac{24}{7} & -\frac{121}{14} \end{bmatrix}$

19. a. Multiply row 2 of A by –5. b. Multiply row 3 of A by –2/5. c. Add 3 times row A to row 2 of A. d. Add 7 times row 2 of A to row 3 of A. e. Exchange rows 1 and 3 of A. f. Subtract 4 times row 3 of A from row 1 of A.

20. a. Subtract 3 times row 4 of A from row 2 of A. b. Exchange rows 2 and 4 of A. c. Multiply row 3 of A by 3/2. d. Multiply row 4 of A by 9. e. Add 5 times row 2 of A to row 4 of A. f. Exchange rows 1 and 4 of A.

27. a. Subtract 3 times column 2 of A from column 4 of A. b. Exchange columns 2 and 4 of A.
c. Multiply column 3 of A by 3/2. d. Multiply column 4 of A by 9.
e. Add 5 times column 1 of A to column 3 of A. f. Exchange columns 1 and 4 of A.

2.8 Exercises

1. a. $A^{-1} = \begin{bmatrix} -3 & \frac{5}{2} \\ 2 & -\frac{3}{2} \end{bmatrix}$; $B^{-1} = \begin{bmatrix} -1 & \frac{4}{3} \\ -2 & \frac{7}{3} \end{bmatrix}$ b. $\begin{bmatrix} \frac{17}{3} & -\frac{9}{2} \\ \frac{32}{3} & -\frac{17}{2} \end{bmatrix}$

c. $\begin{bmatrix} 51 & -27 \\ 64 & -34 \end{bmatrix}$ d. $\begin{bmatrix} \frac{17}{3} & -\frac{9}{2} \\ \frac{32}{3} & -\frac{17}{2} \end{bmatrix}$

7. $A^{-1} = BX^{-1}$ and $B^{-1} = X^{-1}A$. 9. B^{-1} is obtained from A^{-1} by exchanging columns 1 and 3 of A^{-1}, followed by exchanging columns 2 and 5.

10. a. $B = \begin{bmatrix} 5 & -2 & 1 & 0 & 0 \\ -4 & 0 & 7 & 0 & 0 \\ 3 & -9 & -8 & 0 & 0 \\ 0 & 0 & 0 & 3 & -7 \\ 0 & 0 & 0 & -2 & 4 \end{bmatrix}$; $C = \begin{bmatrix} 3 & -7 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 3 & -7 \\ 0 & 0 & -2 & 4 \end{bmatrix}$

b. The entries don't match because the matrices are in opposite locations.

c. $A_2 \oplus A_3 = \begin{bmatrix} 5 & -2 & 1 & 0 & 0 \\ -4 & 0 & 7 & 0 & 0 \\ 3 & -9 & -8 & 0 & 0 \\ 0 & 0 & 0 & -4 & 5 \\ 0 & 0 & 0 & 7 & -3 \end{bmatrix}$;

$$\begin{array}{c}(A_1 \oplus A_2) \oplus A_3 \\ = A_1 \oplus (A_2 \oplus A_3)\end{array} = \begin{bmatrix} 3 & -7 & 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -2 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 & 7 & 0 & 0 \\ 0 & 0 & 3 & -9 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 7 & -3 \end{bmatrix};$$

d. $\begin{bmatrix} 8 & -2 & -1 & 0 & 0 & 0 \\ 4 & 6 & -7 & 0 & 0 & 0 \\ -3 & 5 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & -2 & -5 \end{bmatrix}$; 6×6 e. Only B, with blocks $B_1 = \begin{bmatrix} 3 & -7 \\ -2 & 4 \end{bmatrix}$

and $B_2 = \begin{bmatrix} 8 & -1 \\ 0 & 5 \end{bmatrix}$.

11. a. $A = \begin{bmatrix} 7 & 12 \\ -3 & -5 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} -5 & -12 \\ 3 & 7 \end{bmatrix}$. b. $A = \begin{bmatrix} a & b \\ y & x \end{bmatrix}$, and

$A^{-1} = \begin{bmatrix} x & -b \\ -y & a \end{bmatrix}$

both have only integer entries.

c. $\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}$, with inverse $\begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix}$ d. $\begin{bmatrix} 3 & -7 \\ -7 & 16 \end{bmatrix}$, with inverse

$\begin{bmatrix} -16 & -7 \\ -7 & -3 \end{bmatrix}$.

Other answers are possible by switching entries.

2.9 Exercises

1. a. lower triangular. b. all of the above. c. symmetric. d. all of the above. e. all of the above.

2. a. $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$ b. Symmetric 3. a. $\begin{bmatrix} -12 & -21 & 9 & -6 & 0 \\ 18 & -4 & 2 & 8 & 12 \\ -35 & -21 & -14 & 63 & 7 \end{bmatrix}$ b. $\begin{bmatrix} -27 & -10 & -14 \\ -3 & 8 & 21 \end{bmatrix}$

9. a. $\begin{bmatrix} 24 & 27 & -43 \\ 0 & -6 & 8 \\ 0 & 0 & 28 \end{bmatrix}$ 12. a. $\begin{aligned} T(\vec{e}_1) &= 3\vec{e}_1, \\ T(\vec{e}_2) &= -5\vec{e}_1 + 2\vec{e}_2, \text{ and} \\ T(\vec{e}_3) &= 4\vec{e}_1 + \vec{e}_2 - 7\vec{e}_3. \end{aligned}$ c. $\begin{bmatrix} 1/3 & 5/6 & 13/42 \\ 0 & 1/2 & 1/14 \\ 0 & 0 & -1/7 \end{bmatrix}$

b. $\vec{v}_1 = \frac{1}{3}\vec{e}_1, \vec{v}_2 = \frac{5}{6}\vec{e}_1 + \frac{1}{2}\vec{e}_2, \vec{v}_3 = \frac{13}{42}\vec{e}_1 + \frac{1}{14}\vec{e}_2 - \frac{1}{7}\vec{e}_3.$

Chapter Three Exercises

3.1 Exercises

5. There are no negatives for the vectors, even though there is a zero vector.
6. Not closed under scalar multiplication.
7. Not closed under addition: for example, identity plus its negative yields zero matrix, which is not invertible.
8. $-3 \odot \langle 5,-2\rangle = \langle -15,-2\rangle$. All Axioms are valid except for Axiom 7, so this is not a vector space.
9. $-3 \odot \langle 5,-2\rangle = \langle 15,-6\rangle$. All Axioms are valid except for Axioms 9 and 10, so this is not a vector space.
10. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 7,4\rangle$. All Axioms are valid except for Axioms 7 and 8, so this is not a vector space.
11. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 9,-9\rangle$. Invalid axioms: 3, 4, 5, 6, 7 and 8; not a vector space.
12. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle -9,-3\rangle$. Invalid axioms: 4, 5, 6 and 7; not a vector space.
13. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 13,-1\rangle$. Invalid axioms: 3, 4, 5, 6, and 7; not a vector space.
14. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 9,6\rangle$ and $-3 \odot \langle 5,-2\rangle = \langle -15,12\rangle$. Invalid axioms: 4, 5, 6, 7, 9 and 10; not a vector space.
15. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 9,6\rangle$ and $-3 \odot \langle 5,-2\rangle = \langle -30,6\rangle$. Invalid axioms: 4, 5, 6, 7, 9 and 10; not a vector space.
16. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 3,9\rangle$ and $-3 \odot \langle 5,-2\rangle = \langle 6,-15\rangle$. Invalid axioms: 4, 5, 6, 7, 9 and 10; not a vector space.
17. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle -9,-3\rangle$ and $-3 \odot \langle 5,-2\rangle = \langle -15,-6\rangle$. Invalid axioms: 4, 5, 6, 7, 9 and 10; not a vector space.
18. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 9,0\rangle$ and $-3 \odot \langle 5,-2\rangle = \langle -15,0\rangle$. Invalid axioms: 5, 6, and 10; not a vector space.
19. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 7,6\rangle$ and $-3 \odot \langle 5,-2\rangle = \langle -13,3\rangle$. Invalid axioms: 8, 9, and 10; not a vector space.
    However, there is a zero vector and negatives: $\vec{0}_V = \langle 2,-3\rangle$ and $-\langle x_1, y_1\rangle = \langle 4-x_1,-6-y_1\rangle$.
20. This is a vector space!

3.2 Exercises

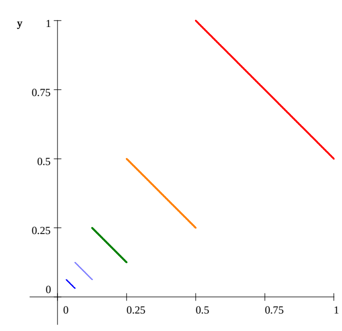
1. Yes, a member. $-7+19x-47x^2 = 3(6+3x-4x^2)-5(5-2x+7x^2)$
2. Yes, a member. $105-28x+39x^2+9x^3 = 7(2-4x+5x^3)+13(7+3x^2-2x^3)$
3. Yes, a member. $\dfrac{2x^2-7x-10}{x^3} = \dfrac{2}{x} - \dfrac{7}{x^2} - \dfrac{10}{x^3}$ 4. Not a member. 5. Yes, a member. $\dfrac{4x+25}{(x+1)(x-2)} = \dfrac{11}{x-2} - \dfrac{7}{x+1}$. 6. Not a member. 7. $\dfrac{x}{x^2-1} = \dfrac{1}{2(x-1)} + \dfrac{1}{2(x+1)}$
10. dependent 11. independent 12. dependent 13. independent 14. dependent 15. dependent 16. dependent
17. dependent 18. independent 19. independent 20. independent 21. independent 22. dependent 23. dependent

24. dependent 25. independent 26. dependent 27. independent 28. dependent 29. dependent 30. dependent
31. dependent 32. dependent 33. dependent 34. independent. 39. d. independent

3.3 Exercises

4. a. $E(x) = \{x^{2n} \mid n \in \mathbb{N}\}$ 5. $O(x) = \{x^{2n+1} \mid n \in \mathbb{N}\}$ 6. a. $S = \left\{ \frac{1}{x^{n+1}} \mid n \in \mathbb{N} \right\}$ 7. a. $(0, \infty)$
d. $f(x) = 1^x = 1$ is a legitimate (constant) function, and we do not care if the functions in S are one-to-one or not.
8. a. $S = \left\{ x^{\frac{1}{n+2}} \mid n \in \mathbb{N} \right\}$
9. independent 10. dependent (the logarithm requires a positive base $b \neq 1$). 11. independent
12. dependent; $S \subset \mathbb{P}^n$, so once you have $n+1$ of these functions, they are definitely dependent; on the other hand, the set S in Exercise 11 is not contained in a single $\mathbb{P}^n$ because there is a polynomial of any degree n in that S.
13. independent; take a limit at a vertical asymptote to show that the coefficient for that term must be 0.
14. independent 15. independent 16. independent 17. independent 18. independent 19. dependent (check out first six vectors) 20. independent
27. a. 1/6, −1/6, 7/6, −7/6, 11/6, −11/6, 13/6, 1/7, −1/7, 2/7, −2/7, 3/7, −3/7, 4/7, 1/8, −1/8, 3/8, −3/8, 5/8, −5/8, 7/8
b. 1/4, −1/3, 3/2, 2, −2, −3/2, 2/3, −1/4, 1/5, 1/6, −1/5, 3/4, −2/3, 5/2, 3, −3, −5/2, 4/3, −3/4, 2/5, −1/6, −1/7,
1/8, −1/7, 5/6. c. $k = i + j - 1$.
28. a. $f(x) = (b-a)x + a$ f. $f(x) = x - a$ h. $f(x) = -x + b$ k. $f(x) = -x + 1$ l. $f(x) = -(x-a) + b = -x + a + b$

m. $$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ -x + \frac{3}{2} & \text{if } x \in \left(\frac{1}{2}, 1\right] \\ -x + \frac{3}{4} & \text{if } x \in \left(\frac{1}{4}, \frac{1}{2}\right] \\ -x + \frac{3}{8} & \text{if } x \in \left(\frac{1}{8}, \frac{1}{4}\right] \\ \vdots & \vdots \\ -x + \frac{3}{2^{n+1}} & \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] \\ \vdots & \vdots \end{cases}$$

n. 

Note: the top of each line segment should be an open hole, and the bottom should be a solid dot, and the graph keeps following the pattern as we get closer to the origin, where $f(0) = 0$.

3.4 Exercises

2. Yes, because every diagonal matrix is also symmetric.
3. A possible basis is $\{\langle 1, 1, \ldots, 1 \rangle\}$. The subspace is 1-dimensional.
4. Yes, it is a 1-dimensional subspace, with possible basis $\{\langle 1, 2, \ldots, n \rangle\}$
5. No. It is not closed under either addition or scalar multiplication, although it does contain the zero vector.
6. A possible basis for this 2-dimensional subspace is $\{1, -24x - 9x^2 + 5x^3\}$.
7. A possible basis for this 2-dimensional subspace is $\{-5 - 7x + 8x^2, 19 - 17x + 2x^3\}$.
8. A possible basis for this 2-dimensional subspace is $\{1 + 2x, -4 + x^2\}$.
9. A possible basis for this 2-dimensional subspace is $\{-1 + 2x, -1 + x^2\}$.
10. A possible basis for this 3-dimensional subspace is $\{-2 + x, 13 - 3x^2, -10 + x^3\}$.
11. It does not contain the zero vector.
12. A possible basis for this 1-dimensional subspace is $\{22 - 10x + x^2 + x^3\}$.
13. A possible basis for this 1-dimensional subspace is $\{2e^{2x} - 3e^{3x} + e^{5x}\}$.
14. A possible basis for this 2-dimensional subspace is $\{-2e^{2x} + e^{3x}, -4e^{2x} + e^{5x}\}$. W_1 is a subspace of W_2.
15. A possible basis for this 2-dimensional subspace is:
$\left\{\left(\sqrt{2} - 1\right)\sin(x) + \cos(x), -\sqrt{2}\sin(x) + \tan(x)\right\}$
16. c. It doesn't contain the zero function $z(x)$.
17. Another hint: think of the ***factors*** of such a member of W. The subspace is 1-dimensional.
18. The sum of $p(x) = x + 2$ and $q(x) = x - 3$, which are both in W, is $r(x) = 2x - 1$, which is not in W. Can you come up with a counterexample where both p and q are quadratics?
19. Yes. 20. Yes. 21. Yes. 22. Yes. 23. No. The set is dependent, even though S is a subset of W.
24. Yes. 25. No. This polynomial is not in W. 26. Yes. 27. Yes. 32. 2-dim 36. $dim(Diag(n)) = n$.
38. $dim(Upper(n)) = n(n+1)/2$. 39. The transpose of the basis vectors you found in Exercise 38 will form a basis for $Lower(n)$, so the two spaces have exactly the same dimension.
41. The basis should have two kinds of matrices: those which are all 0 except for a single 1 on the main diagonal (thus there are n of these), and those which are all 0 except for a single 1 in row i, column j, as well as in row j, column i, where $i \neq j$. There are $1 + 2 + \cdots + (n-1)$ of these. Thus there are $1 + 2 + \cdots + (n-1) + n = n(n+1)/2$ members of this basis, which is $dim(Sym(n))$.
42. b. Possible answer: $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$; $dim(Bisym(2)) = 2$

d. Possible answer: $\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$;
$dim(Bisym(3)) = 4$.

e. $\begin{bmatrix} a & b & c & d \\ b & e & f & c \\ c & f & e & b \\ d & c & b & a \end{bmatrix}$; $dim(Bisym(4)) = 6$ f. $\begin{bmatrix} a & b & c & d & e \\ b & f & g & h & d \\ c & g & i & g & c \\ d & h & g & f & b \\ e & d & c & b & a \end{bmatrix}$;

$dim(Bisym(5)) = 9$

Use one matrix for every distinct letter.

43. d. Possible answer: $\left\{ I_2, \begin{bmatrix} 1 & 5 \\ -7 & 0 \end{bmatrix} \right\}$; it is 2-dimensional.

45. e. Use the Ordinary Comparison Test.
f. D does not contain the zero vector (zero series), which is absolutely convergent.
g. It is not closed under vector addition.

3.5 Exercises

1. a. 21. b. $\sqrt{3}$ c. 1 2. a. $\langle 0, 3/5, 1/2 \rangle$ b. $\langle 1, 7/25, 1/2 \rangle$ c. $\left\langle 0, 3/4, 1/\sqrt{3} \right\rangle$
3. a. $\langle -66, 6 \rangle$ b. $\{(x+3)(x-1)\}$ or $\{x^2 + 2x - 3\}$. 4. a. $\langle -996, 156, -84 \rangle$ b. $\{(x+5)(x-3)(x+2)\}$
5. a. $\langle 117, 13, 18 \rangle$ b. $\{z(x)\}$ 6. a. $\langle 6, 28, -26 \rangle$ 7. a. $\langle -33, -2, -10, 16/3 \rangle$ 8. a. $12x + 10$
9. a. $3x^4 + 2x^3 - 7x^2$ 10. a. $x^3 + x^2 - 7x$ 11. a. $-5e^{-x} - 6e^{2x}$ d. $\{z(x)\}$ e. $range(D) = W$.
12. a. $7e^x \sin(x) + e^x \cos(x)$ d. $\{z(x)\}$ e. $range(D) = W$.
13. a. $3e^{-3x} \sin(2x) + 37e^{-3x} \cos(2x)$ d. $\{z(x)\}$ e. $range(D) = W$.
14. a. $33e^{5x} - 10xe^{5x}$ d. $\{z(x)\}$ e. $range(D) = W$.
15. a. $20x^2 e^{-4x} - 18xe^{-4x} + 30e^{-4x}$ d. $\{z(x)\}$ e. $range(D) = W$.
16. a. $-4\ln 5x^2 \cdot 5^x + (9(\ln 5) - 8)x \cdot 5^x + (9 - 2(\ln 5))5^x$ d. $\{z(x)\}$ e. $range(D) = W$.
17. a. $6x^2 - 16x + 3$ d. $\{1\}$ e. $\{1, x, x^2\}$.
18. a. $-18x\sin(2x) + 8x\cos(2x) - 12\sin(2x) - \cos(2x)$ d. $\{z(x)\}$ e. $range(D) = W$.
19. a. $27\sin(x) - \cos(x)$ 20. a. $120e^{4x}\sin(3x) + 102e^{4x}\cos(3x)$
21. a. $(ac_1 - bc_2)e^{ax}\sin bx + (ac_2 + bc_1)e^{ax}\cos bx$
22. a. $-4c_1e^{-4x} + 3c_2e^{3x} + 5c_3e^{5x}$ d. $91c_1e^{-4x} + 64c_3e^{5x}$ e. $\{e^{3x}\}$ f. $\{e^{-4x}, e^{5x}\}$ 26. a. $\begin{bmatrix} 4 & 0 \\ -3 & 1 \\ 5 & -7 \end{bmatrix}$

3.6 Exercises

1. a. $\langle -13/2, 19/2, 8 \rangle$ c. $\langle -1/2, 1/2, 1 \rangle$.
2. b. $\langle 3/2, 27/2, 83, -545/3 \rangle$
3. a. $\left\langle -1/\sqrt{2}, 1/\sqrt{2} \right\rangle$ b. $\langle 4/5, 3/5 \rangle$ c. $\langle -12/13, 5/13 \rangle$ d. $\langle 20/29, 21/29 \rangle$

4. a. $\begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & 1 \end{bmatrix}$ c. $\langle 82, 6 \rangle$ 5. a. $\begin{bmatrix} 1 & -5 & 25 & -125 \\ 1 & 3 & 9 & 27 \\ 1 & -2 & 4 & -8 \end{bmatrix}$ c. $\langle -1285, 179, -91 \rangle$

6. a. $\begin{bmatrix} 1 & -5 & 25 \\ 1 & 3 & 9 \\ 1 & -2 & 4 \end{bmatrix}$ c. $\langle 91, 27, 22 \rangle$ 7. a. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 2 & 1 & 8 \end{bmatrix}$ c. $\langle 6, -33, 59 \rangle$

8. a. $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 1 & 1/2 & 1/3 \end{bmatrix}$ c. $\langle 42, 9, 14, 23/6 \rangle$ 9. a. $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ c. $42x - 16$

10. a. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$ c. $\frac{7}{3}x^2 - \frac{5}{2}x^2 + 4x$ 11. a. $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ b. $-5e^{-x} - 6e^{2x}$

12. a. $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ b. $7e^x \sin(x) + e^x \cos(x)$ 13. a. $\begin{bmatrix} -3 & -2 \\ 2 & -3 \end{bmatrix}$

b. $3e^{-3x} \sin(2x) + 37e^{-3x} \cos(2x)$

14. a. $\begin{bmatrix} 5 & 0 \\ 1 & 5 \end{bmatrix}$ b. $-10xe^{5x} + 33e^{5x}$

15. a. $\begin{bmatrix} -4 & 0 & 0 \\ 2 & -4 & 0 \\ 0 & 1 & -4 \end{bmatrix}$ b. $20x^2e^{-4x} - 18xe^{-4x} + 30e^{-4x}$

16. $\begin{bmatrix} \ln(5) & 0 & 0 \\ 2 & \ln(5) & 0 \\ 0 & 1 & \ln(5) \end{bmatrix}$ b. $-4\ln(5)x^2 \cdot 5^x + (9\ln(5) - 8)x \cdot 5^x + (-2\ln(5) + 9)5^x$

17. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ b. $6x^2 - 16x + 3$ 18. $\begin{bmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

b. $-18x\sin(2x) + 8x\cos(2x) - 12\sin(2x) - \cos(2x)$

19. b. $\begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix}$ 20. b. $[D]_B = Diag(k_1, k_2, \ldots, k_n)$ c. a diagonal matrix

21. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ 22. $\begin{bmatrix} k & 0 & 0 \\ 2 & k & 0 \\ 0 & 1 & k \end{bmatrix}$ c. $kx^n e^{kx} + nx^{n-1}e^{kx}$

23. a. $\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$ b. $27\sin(x) - \cos(x)$ c. $\frac{13}{5}\sin(x) - \frac{9}{5}\cos(x)$

24. a. $\begin{bmatrix} -3 & -15 \\ 15 & -3 \end{bmatrix}$ c. $-96e^{4x}\sin(3x) - 66e^{4x}\cos(3x)$

25. a. $45x^2 + 6x - 20$ d. $\begin{bmatrix} 2 & -1 & 4 & -2 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 9 \end{bmatrix}$

26. a. $-11x^3 - 36x^2 + 60x - 41$ d. $\begin{bmatrix} -5 & 3 & 0 \\ 2 & -5 & 6 \\ 0 & 3 & -5 \\ -1 & 2 & 0 \end{bmatrix}$

27. a. $\langle 95, -15, -6\rangle$. b. $365x - 211$. c. $T(1) = 4x - 2$, $T(x) = 21x - 7$, and $T(x^2) = 66x - 36$.

d. $[T]_{S,S'} = \begin{bmatrix} -2 & -7 & -36 \\ 4 & 21 & 66 \end{bmatrix}$

28. a. $\langle -11, 3\rangle$ b. $-46x^2 + 63x + 126$ c. $T(1) = 5x^2 - 6x - 9$; $T(x) = -7x^2 + 11x + 27$ d. $\begin{bmatrix} -9 & 27 \\ -6 & 11 \\ 5 & -7 \end{bmatrix}$

29. a. $\langle 69/2, -14, -3\rangle$. b. $\frac{311}{2} - 167x + \frac{59}{2}x^2$. c. $T(1) = \frac{9}{2} - 3x + \frac{1}{2}x^2$, $T(x) = \frac{25}{2} - 10x + \frac{3}{2}x^2$, and $T(x^2) = \frac{83}{2} - 46x + \frac{17}{2}x^2$. d. $\begin{bmatrix} 9/2 & 25/2 & 83/2 \\ -3 & -10 & -46 \\ 1/2 & 3/2 & 17/2 \end{bmatrix}$

31. $[proj_\Pi] = \frac{1}{122}\begin{bmatrix} 113 & -21 & 24 \\ -21 & 73 & 56 \\ 24 & 56 & 58 \end{bmatrix}$; $[refl_\Pi] = \frac{1}{61}\begin{bmatrix} 52 & -21 & 24 \\ -21 & 12 & 56 \\ 24 & 56 & -3 \end{bmatrix}$;

$[proj_L] = \frac{1}{122}\begin{bmatrix} 9 & 21 & -24 \\ 21 & 49 & -56 \\ -24 & -56 & 64 \end{bmatrix}$

32. $[proj_\Pi] = \frac{1}{83}\begin{bmatrix} 58 & 15 & -35 \\ 15 & 74 & 21 \\ -35 & 21 & 34 \end{bmatrix}$; $[refl_\Pi] = \frac{1}{83}\begin{bmatrix} 33 & 30 & -70 \\ 30 & 65 & 42 \\ -70 & 42 & -15 \end{bmatrix}$;

$[proj_L] = \frac{1}{83}\begin{bmatrix} 25 & -15 & 35 \\ -15 & 9 & -21 \\ 35 & -21 & 49 \end{bmatrix}$

33. $[proj_\Pi] = \frac{1}{30}\begin{bmatrix} 26 & 2 & -10 \\ 2 & 29 & 5 \\ -10 & 5 & 5 \end{bmatrix}$; $[refl_\Pi] = \frac{1}{15}\begin{bmatrix} 11 & 2 & -10 \\ 2 & 14 & 5 \\ -10 & 5 & -10 \end{bmatrix}$;

$[proj_L] = \frac{1}{30}\begin{bmatrix} 4 & -2 & 10 \\ -2 & 1 & -5 \\ 10 & -5 & 25 \end{bmatrix}$

34. a. choose $\langle 2,0,3\rangle$ and $\langle 0,1,0\rangle$ (note that the 2nd vector satisfies the equation);

f. $[proj_\Pi] = \frac{1}{13}\begin{bmatrix} 4 & 0 & 6 \\ 0 & 13 & 0 \\ 6 & 0 & 9 \end{bmatrix}$ h. $[refl_\Pi] = \frac{1}{13}\begin{bmatrix} -5 & 0 & 12 \\ 0 & 13 & 0 \\ 12 & 0 & 5 \end{bmatrix}$

i. $[proj_L] = \frac{1}{13}\begin{bmatrix} 9 & 0 & -6 \\ 0 & 0 & 0 \\ -6 & 0 & 4 \end{bmatrix}$

35. d. $C = \begin{bmatrix} -c & 0 & a \\ 0 & 1 & 0 \\ a & 0 & c \end{bmatrix}$ is one possible answer. 37. $S' = \{\vec{w}_1, \vec{w}_2, \vec{w}_4\}$; $\vec{w}_3 = 4\vec{w}_1 - 3\vec{w}_2$

38. $S' = \{\vec{w}_1, \vec{w}_2, \vec{w}_4\}$; $\vec{w}_3 = -4\vec{w}_1 + 3\vec{w}_2$; $\vec{w}_5 = 2\vec{w}_1 - 5\vec{w}_2 + 7\vec{w}_4$

39. $S' = \{\vec{w}_1, \vec{w}_2, \vec{w}_5\}$; $\vec{w}_3 = 4\vec{w}_1 + 9\vec{w}_2$; $\vec{w}_4 = 5\vec{w}_1 + 8\vec{w}_2$; $\vec{w}_6 = -3\vec{w}_1 + 4\vec{w}_2 - 7\vec{w}_5$

40. $S' = \{\vec{w}_1, \vec{w}_2, \vec{w}_4\}$; $\vec{w}_3 = 4\vec{w}_1 - 3\vec{w}_2$; $\vec{w}_5 = 6\vec{w}_1 - 3\vec{w}_2 - 4\vec{w}_4$

42. c. $[S_{\vec{u}}]_{B,B'} = \begin{bmatrix} 0 & -a/c \\ 1 & -b/c \end{bmatrix}$ d. $\begin{bmatrix} 0 & -3/5 \\ 1 & 2/5 \end{bmatrix}$

3.7 Exercises

1. a. No. b. Yes, because $dim(\mathbb{P}^2) < dim(\mathbb{R}^4)$.

   c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

   d. $ker(T) = \{z(x)\}$, so it has no basis, and $nullity(T) = 0$.
   e. $range(T)$ has basis $\{\langle 1,0,0,1\rangle, \langle -2,1,0,1/2\rangle, \langle 4,2,2,1/3\rangle\}$, and $rank(T) = 3$
   f. T is one-to-one but not onto. g. $3 + 0 = 3 = dim(\mathbb{P}^2)$
   h. $p(x) = 4 - 7x + 5x^2$ is the only such polynomial.
2. a. Yes, because $dim(\mathbb{P}^3) > dim(\mathbb{P}^1)$. b. No.

   c. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   d. $ker(T)$ has basis $\{1, x\}$ and $nullity(T) = 2$.
   e. $range(T)$ has basis $\{x^2, x^3\}$ and $rank(T) = 2$.
   f. T is neither one-to-one nor onto. g. $2 + 2 = 4 = dim(\mathbb{P}^3)$
3. a. No. b. Yes, because $dim(\mathbb{P}^2) < dim(\mathbb{P}^3)$.

   c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

   d. $ker(T) = \{z(x)\}$, so it has no basis and $nullity(T) = 0$.
   e. $range(T)$ has basis $\{x, x^2, x^3\}$ (we can clear the fractions) and $rank(T) = 3$.
   f. T is one-to-one but not onto. g. $0 + 3 = 3 = dim(\mathbb{P}^2)$.
4. a. Yes, because $dim(\mathbb{P}^3) > dim(\mathbb{P}^2)$. b. No.

   c. $\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   d. $ker(T)$ has basis $\{1 + 2x\}$ and $nullity(T) = 1$.
   e. $range(T)$ has basis $\{2, 4 + 4x, -2 + 6x + 9x^2\}$ or $\{1, x, x^2\}$; either basis is acceptable because $rank(T) = 3$.
   f. T is not one-to-one but T is onto. g. $3 + 1 = 4 = dim(\mathbb{P}^3)$
5. a. No. b. Yes, because $dim(\mathbb{P}^2) < dim(\mathbb{P}^3)$.

   c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

d. $ker(T) = \{z(x)\}$, so it has no basis and $nullity(T) = 0$.
e. $range(T)$ has basis $\{-5 + 2x - x^3, 3 - 5x + 3x^2 + 2x^3, 6x - 5x^2\}$ and $rank(T) = 3$.
f. T is one-to-one but not onto. g. $3 + 0 = dim(\mathbb{P}^2)$.

6. b. No. c. Yes, because $dim(\mathbb{P}^2) < dim(\mathbb{P}^3)$.

d. $\begin{bmatrix} 0 & -5 & -8 \\ 0 & 0 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ e. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

f. $ker(T)$ has basis $\{1\}$ and $nullity(T) = 1$.
g. $range(T)$ has basis $\{-5 + x^2, -8 - 6x + 4x^3\}$ and $rank(T) = 2$.
h. T is neither one-to-one nor onto. i. $2 + 1 = 3 = dim(\mathbb{P}^2)$.

7. b. Yes, because $dim(\mathbb{P}^3) > dim(\mathbb{P}^2)$. c. No.

d. $\begin{bmatrix} 6 & -3 & 6 & -21 \\ -10 & 5 & -10 & 35 \\ 2 & -1 & 2 & -7 \end{bmatrix}$ e. $\begin{bmatrix} 1 & -1/2 & 1 & -7/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

f. $ker(T)$ has basis $\{1 + 2x, -1 + x^2, 7 + 2x^3\}$ and $nullity(T) = 3$.
g. $range(T)$ has basis $\{6 - 10x + 2x^2\}$ and $rank(T) = 1$.
h. T is neither one-to-one nor onto. i. $1 + 3 = 4 = dim(\mathbb{P}^3)$.

8. a. Yes, because $dim(\mathbb{P}^2) > dim(\mathbb{P}^1)$. b. No.

c. $\begin{bmatrix} 1 & 0 & \frac{2}{7} \\ 0 & 1 & -\frac{27}{7} \end{bmatrix}$

d. $ker(T)$ has basis $\{147 - 6x - 7x^2\}$ and $nullity(T) = 1$
e. $range(T)$ has basis $\{x + 3, 2x - 1\}$ or $\{1, x\}$; either basis is acceptable because $rank(T) = 2$.
f. T is not one-to-one but it is onto. g. $2 + 1 = 3 = dim(\mathbb{P}^2)$.

9. a. No. b. Yes, because $dim(\mathbb{P}^1) < dim(\mathbb{P}^2)$.

c. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

d. $ker(T) = \{z(x)\}$, so it has no basis and $nullity(T) = 0$.
e. $range(T) = Span(\{5x^2 - 6x - 9, 3x^2 - x + 9\})$ and $rank(T) = 2$.
f. T is one-to-one but not onto. g. $2 + 0 = 2 = dim(\mathbb{P}^1)$.

10. a. Yes, because $dim(\mathbb{P}^2) > dim(\mathbb{P}^1)$. b. No.

c. $\begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$

d. $ker(T)$ has basis $\{2x + 5, -2x^2 + 2x - 3\}$ and $nullity(T) = 2$
e. $range(T)$ has basis $\{3x - 7\}$ and $rank(T) = 1$.

f. T is neither one-to-one nor onto. g. $1 + 2 = 3$.

11. a. No. b. Yes, because $dim(\mathbb{P}^1) < dim(\mathbb{P}^2)$.

c. $\begin{bmatrix} 1 & 5/7 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

d. $ker(T) = \{-7x + 2\}$, and $nullity(T) = 1$.

e. $range(T)$ has basis $\{2x^2 + x + 8\}$ and $rank(T) = 1$.

f. T is neither one-to-one nor onto. g. $1 + 1 = 2 = dim(\mathbb{P}^1)$.

12. a. No. b. No.

c. $\begin{bmatrix} 1 & 0 & -\frac{27}{11} \\ 0 & 1 & \frac{14}{11} \\ 0 & 0 & 0 \end{bmatrix}$

d. $ker(T)$ has basis $\{27 - 14x + 11x^2\}$, and $nullity(T) = 1$.

e. $range(T)$ has basis $\{4 - x + 5x^2, 3 + 2x + 12x^2\}$, and $rank(T) = 2$.

h. $p(x) = 3 - 2x + \frac{c_2}{11}(27 - 14x + 11x^2)$ ($\frac{c_2}{11}$ can be replaced by c)

13. b. $[T_1]_{B,B'} = \begin{bmatrix} 4 & -5 & 0 \\ 0 & 7 & -10 \\ 0 & 1 & 10 \\ 0 & 0 & 2 \end{bmatrix}$, and $[T_2]_{B',B} = \begin{bmatrix} 0 & 3 & -10 & 0 \\ 0 & 0 & 6 & -30 \\ 0 & 0 & 0 & 9 \end{bmatrix}$.

c. The codomain of the first is the same as the domain of the second, in either order.

d. $[T_2 \circ T_1]_{B,B} = \begin{bmatrix} 0 & 11 & -130 \\ 0 & 6 & 0 \\ 0 & 0 & 18 \end{bmatrix}$ and $[T_1 \circ T_2]_{B',B'} = \begin{bmatrix} 0 & 12 & -70 & 150 \\ 0 & 0 & 42 & -300 \\ 0 & 0 & 6 & 60 \\ 0 & 0 & 0 & 18 \end{bmatrix}$

14. b. $[T_1]_{B,B'} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ and $[T_2]_{B',B''} = \begin{bmatrix} 1 & -3 & 9 & -27 \\ 0 & 1 & 4 & 12 \\ 0 & 0 & 2 & -6 \end{bmatrix}$.

c. $[T_2 \circ T_1]_{B,B''} = \begin{bmatrix} -3 & 9 & -27 \\ 2 & 11 & 36 \\ 0 & 4 & -6 \end{bmatrix}$.

d. No, because the codomain of T_2, which is $\mathbb{R}^3$, is not the domain of T_1, which is $\mathbb{P}^2$. The two spaces $\mathbb{R}^3$ and $\mathbb{P}^2$ are both 3-dimensional, but the ***composition*** $T_1 \circ T_2$ is still undefined. e. Yes, the ***matrix product*** $[T_1]_{B,B'} \cdot [T_2]_{B',B''}$ is a well-defined 4×4 matrix. However, it is completely meaningless in this case.

15. a. No. b. Yes; domain $\mathbb{P}^2$ and codomain $\mathbb{P}^1$. c. $10x^3 - 2x^2 + 16x + 11$

d. $36x - 167$ e. $\begin{bmatrix} 19 & 25 & 33 \\ 2 & -12 & 62 \end{bmatrix}$

16. a. Yes; domain $\mathbb{P}^2$ and codomain $\mathbb{P}^2$. b. Yes; domain $\mathbb{P}^1$ and codomain $\mathbb{P}^1$.
c. $35x^2 - 127x - 11$ d. $41x + 7$
e. $\begin{bmatrix} 11 & -14 \\ 13 & 3 \end{bmatrix}$ g. $220x - 245$ h. $1540x^2 - 5525x - 295$ i. $\begin{bmatrix} -13 & 9 & -16 \\ 32 & -1 & 14 \\ 49 & -7 & 28 \end{bmatrix}$

17. a. $[D^2]_B = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ and $[D^3]_B = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}$; b. $f''(x) = 5e^{-x} - 12e^{2x}$;
$f'''(x) = -5e^{-x} - 24e^{2x}$

18. a. $[D^2]_B = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ and $[D^3]_B = \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}$;
b. $f''(x) = 5e^{-x} - 12e^{2x}$; $f'''(x) = -2e^x\sin(x) + 14e^x\cos(x)$

19. a. $[D^2]_B = \begin{bmatrix} 5 & 12 \\ -12 & 5 \end{bmatrix}$ and $[D^3]_B = \begin{bmatrix} 9 & -46 \\ 46 & 9 \end{bmatrix}$;
b. $f''(x) = -83e^{-3x}\sin(2x) - 105e^{-3x}\cos(2x)$;
$f'''(x) = 459e^{-3x}\sin(2x) + 149e^{-3x}\cos(2x)$

20. a. $[D^2]_B = \begin{bmatrix} 25 & 0 \\ 10 & 25 \end{bmatrix}$ and $[D^3]_B = \begin{bmatrix} 125 & 0 \\ 75 & 125 \end{bmatrix}$
b. $f''(x) = -50xe^{5x} + 155e^{5x}$; $f'''(x) = -250xe^{5x} + 725e^{5x}$

21. a. $[D^2]_B = \begin{bmatrix} 16 & 0 & 0 \\ -16 & 16 & 0 \\ 2 & -8 & 16 \end{bmatrix}$ and $[D^3]_B = \begin{bmatrix} -64 & 0 & 0 \\ 96 & -64 & 0 \\ -24 & 48 & -64 \end{bmatrix}$
b. $f''(x) = -80x^2e^{-4x} + 112xe^{-4x} - 138e^{-4x}$; $f'''(x) = 320x^2e^{-4x} - 608xe^{-4x} + 664e^{-4x}$

22. a. $[D^2]_B = \begin{bmatrix} (\ln(5))^2 & 0 & 0 \\ 4\ln 5 & (\ln(5))^2 & 0 \\ 2 & 2\ln 5 & (\ln(5))^2 \end{bmatrix}$ and
$[D^3]_B = \begin{bmatrix} (\ln(5))^3 & 0 & 0 \\ 6(\ln(5))^2 & (\ln(5))^3 & 0 \\ 6\ln 5 & 3(\ln(5))^2 & (\ln(5))^3 \end{bmatrix}$
b.
$f''(x) = -4(\ln(5))^2x^25^x + \left(9(\ln(5))^2 - 16\ln(5)\right)x5^x + \left(-2(\ln(5))^2 + 18\ln(5) - 8\right)5^x$;
$f'''(x) = -4(\ln(5))^3x^25^x + \left(9(\ln(5))^3 - 24(\ln(5))^2\right)x5^x$
$+ \left(-2(\ln(5))^3 + 27(\ln(5))^2 - 24\ln(5)\right)5^x$

23. a. $[D^2]_B = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & -4 & -4 & 0 \\ 4 & 0 & 0 & -4 \end{bmatrix}$ and $[D^3]_B = \begin{bmatrix} 0 & 8 & 0 & 0 \\ -8 & 0 & 0 & 0 \\ -12 & 0 & 0 & 8 \\ 0 & -12 & -8 & 0 \end{bmatrix}$

b. $f^{//}(x) = -16x\sin(2x) - 36x\cos(2x) - 16\sin(2x) - 16\cos(2x)$;
$f^{///}(x) = 72x\sin(2x) - 32x\cos(2x) + 16\sin(2x) - 68\cos(2x)$

24. a. $[D^2]_B = \begin{bmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{bmatrix}$ and $[D^3]_B = \begin{bmatrix} a^3 - 3ab^2 & b^3 - 3a^2b \\ -b^3 + 3a^2b & a^3 - 3ab^2 \end{bmatrix}$

3.8 Exercises

1. b. $[T]_{B,B^{/}} = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 5 & 25 \\ 0 & 1 & 4 \end{bmatrix}$. c. $[T]_{B,B^{/}}^{-1} = \begin{bmatrix} -\frac{5}{16} & \frac{21}{16} & -\frac{15}{2} \\ -\frac{1}{4} & \frac{1}{4} & -1 \\ \frac{1}{16} & -\frac{1}{16} & \frac{1}{2} \end{bmatrix}$ d. $p(x) = 9 - 7x + 5x^2$.

2. b. $[T]_{B,B^{/}} = \begin{bmatrix} 1 & -4 & 16 & -64 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 0 & 1 & -2 & 3 \end{bmatrix}$. c. $\begin{bmatrix} -\frac{3}{25} & \frac{33}{25} & -\frac{1}{5} & -\frac{6}{5} \\ \frac{19}{175} & -\frac{23}{100} & \frac{17}{140} & \frac{13}{10} \\ \frac{1}{35} & -\frac{1}{10} & \frac{1}{14} & 0 \\ -\frac{3}{175} & \frac{1}{100} & \frac{1}{140} & -\frac{1}{10} \end{bmatrix}$

d. $p(x) = -11 + 7x - 5x^2 + 2x^3$.

3. a. $5x^2 - 9x + 14$ b. $-3x^2 + 4x + 7$
4. a. $9x^2 - 5x + 17$ b. $-8x^2 - 19x + 23$
5. a. $-4x^2 + 9x - 3$ b. $15x^2 - 8x - 11$
6. a. $-5x^3 + 8x^2 - 3x + 11$ b. $-13x^2 + 7x + 11$
7. a. $-4x^3 + 12x^2 + 19x - 7$ b. $17x^3 - 5x^2 + 12x + 8$
8. a. $-9x^3 + 13x^2 - 5x + 11$ b. $4x^3 - 15x + 8$
9. a. $9x^3 + 7x^2 - 11$ b. $11x^3 - 18x + 9$
10. a. $\frac{2}{3}x^3 - 9x^2 - 11x + 17$ b. $-12x^3 + \frac{7}{4}x^2 + 9x - 3$
11. a. $[D]_B^{-1} = \frac{1}{13}\begin{bmatrix} -3 & 2 \\ -2 & -3 \end{bmatrix}$ b. $7e^{-3x}\sin(2x) - 5e^{-3x}\cos(2x) + C$.
12. a. $[D]_B^{-1} = \frac{1}{25}\begin{bmatrix} 5 & 0 \\ -1 & 5 \end{bmatrix}$ b. $3xe^{5x} + 8e^{5x} + C$.
13. a. $[D]_B^{-1} = \frac{1}{32}\begin{bmatrix} -8 & 0 & 0 \\ -4 & -8 & 0 \\ -1 & -2 & -8 \end{bmatrix}$ b. $4x^2e^{-4x} - 9xe^{-4x} - 3e^{-4x} + C$.

14. a. $[D]_B^{-1} = \begin{bmatrix} \frac{1}{\ln 5} & 0 & 0 \\ -\frac{2}{(\ln 5)^2} & \frac{1}{\ln 5} & 0 \\ \frac{2}{(\ln 5)^3} & -\frac{1}{(\ln 5)^2} & \frac{1}{\ln 5} \end{bmatrix}$

b. $\frac{7}{\ln 5}x^2 \cdot 5^x - \left(\frac{14}{(\ln 5)^2} + \frac{4}{\ln 5}\right)x \cdot 5^x + \left(\frac{14}{(\ln 5)^3} + \frac{4}{(\ln 5)^2} + \frac{9}{\ln 5}\right)5^x + C.$

15. a. $[D]_B^{-1} = \frac{1}{4}\begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{bmatrix}$;

b. $3x\sin(2x) - 7x\cos(2x) - 5\sin(2x) + 6\cos(2x) + C$

16. a. $[D]_B^{-1} = \frac{1}{k^2+m^2}\begin{bmatrix} k & m \\ -m & k \end{bmatrix}$

b. $\frac{k}{k^2+m^2}e^{kx}\sin(mx) - \frac{m}{k^2+m^2}e^{kx}\cos(mx) + C$ and $\frac{m}{k^2+m^2}e^{kx}\sin(mx) + \frac{k}{k^2+m^2}e^{kx}\cos(mx) + C.$

17. $f(x) = -2x^2e^{-3x} + 8xe^{-3x} + 3e^{-3x}$

18. a. $W = \mathbb{P}^2$ (use the standard basis) b. $T = 3I_3 + 5D - 2D^2$

c. $[T]_B = \begin{bmatrix} 3 & 5 & -4 \\ 0 & 3 & 10 \\ 0 & 0 & 3 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{1}{27}\begin{bmatrix} 9 & -15 & 62 \\ 0 & 9 & -30 \\ 0 & 0 & 9 \end{bmatrix}$ e. $\frac{1}{3}(2 - 7x + 5x^2)$

19. a. $W = \mathbb{P}^3$ (use the standard basis) b. $T = 3I_4 + 5D - 2D^2$

c. $[T]_B = \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 10 & -12 \\ 0 & 0 & 3 & 15 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{1}{27}\begin{bmatrix} 9 & -15 & 62 & -370 \\ 0 & 9 & -30 & 186 \\ 0 & 0 & 9 & -45 \\ 0 & 0 & 0 & 9 \end{bmatrix}$

e. $-3511 + 1752x - 747x^2 + 162x^3$

20. a. $W = Span(B),\ B = \{\sin(x), \cos(x)\}$ b. $T = -7I_W + 8D + 3D^2$

c. $[T]_B = \begin{bmatrix} -10 & -8 \\ 8 & -10 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{1}{82}\begin{bmatrix} -5 & 4 \\ -4 & -5 \end{bmatrix}$ e. $-12\sin(x) + 7\cos(x)$

21. a. $W = Span(B),\ B = \{\sin(x), \cos(x)\}$ b. $T = 8I_W + 3D - 4D^2 - 2D^3$

c. $[T]_B = \begin{bmatrix} 12 & -5 \\ 5 & 12 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{1}{169}\begin{bmatrix} 12 & 5 \\ -5 & 12 \end{bmatrix}$ e. $5\sin(x) + 7\cos(x)$

22. a. $W = Span(B),\ B = \{\sin(2x), \cos(2x)\}$ b. $T = -7I_W + 8D + 3D^2$

c. $[T]_B = \begin{bmatrix} -19 & -16 \\ 16 & -19 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{1}{617}\begin{bmatrix} -19 & 16 \\ -16 & -19 \end{bmatrix}$ e. $-5\sin(2x) - 14\cos(2x)$

23. a. $W = Span(B),\ B = \{\sin(2x), \cos(2x)\}$ b. $T = 8I_W + 3D - 4D^2 - 2D^3$

c. $[T]_B = \begin{bmatrix} 24 & -22 \\ 22 & 24 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{1}{530}\begin{bmatrix} 12 & 11 \\ -11 & 12 \end{bmatrix}$ e. $3\sin(2x) - 8\cos(2x)$

24. a. $W = Span(B),\ B = \{e^{-3x}\sin(2x), e^{-3x}\cos(2x)\}$ b. $T = 4I_W + 5D - 9D^2$
c. $[T]_B = \begin{bmatrix} -56 & -118 \\ 118 & -56 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{1}{8530}\begin{bmatrix} -28 & 59 \\ -59 & -28 \end{bmatrix}$ e.
$17e^{-3x}\sin(2x) + 11e^{-3x}\cos(2x)$

25. a. $W = Span(B),\ B = \{e^{-3x}\sin(2x), e^{-3x}\cos(2x)\}$ b. $T = -6I_W + 2D + 7D^2 + 3D^3$
c. $[T]_B = \begin{bmatrix} 50 & -58 \\ 58 & 50 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{1}{2932}\begin{bmatrix} 25 & 29 \\ -29 & 25 \end{bmatrix}$ e.
$5e^{-3x}\sin(2x) + 2e^{-3x}\cos(2x)$

26. a. $W = Span(B),\ B = \{xe^{5x}, e^{5x}\}$ b. $T = 4I_W - 9D + 2D^2$
c. $[T]_B = \begin{bmatrix} 9 & 0 \\ 11 & 9 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{1}{81}\begin{bmatrix} 9 & 0 \\ -11 & 9 \end{bmatrix}$ e. $4xe^{5x} - 7e^{5x}$

27. a. $W = Span(B),\ B = \{xe^{5x}, e^{5x}\}$ b. $T = 2I_W - 7D - 3D^2 + 4D^3$
c. $[T]_B = \begin{bmatrix} 64 & 0 \\ 77 & 64 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{1}{4096}\begin{bmatrix} 64 & 0 \\ -77 & 64 \end{bmatrix}$ e. $-9xe^{5x} + 13e^{5x}$

28. a. $W = Span(B),\ B = \{x^2e^{-4x}, xe^{-4x}, e^{-4x}\}$ b. $T = 8I_W + 11D + 3D^2$
c. $[T]_B = \begin{bmatrix} 12 & 0 & 0 \\ -26 & 12 & 0 \\ 6 & -13 & 12 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{1}{864}\begin{bmatrix} 72 & 0 & 0 \\ 156 & 72 & 0 \\ 133 & 78 & 72 \end{bmatrix}$
e. $3x^2e^{-4x} + 7xe^{-4x} + 5e^{-4x}$

29. a. $W = Span(B),\ B = \{x^2e^{-4x}, xe^{-4x}, e^{-4x}\}$ b. $T = 11I_W - 8D + 4D^2 + 3D^3$
c. $[T]_B = \begin{bmatrix} -85 & 0 & 0 \\ 208 & -85 & 0 \\ -64 & 104 & -85 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{-1}{614125}\begin{bmatrix} 7225 & 0 & 0 \\ 17680 & 7225 & 0 \\ 16192 & 8840 & 7225 \end{bmatrix}$
e. $2x^2e^{-4x} + 9xe^{-4x} + 7e^{-4x}$

30. a. $W = Span(B),\ B = \{\sinh(3x), \cosh(3x)\}$ b. $T = -8I_W + 9D + 4D^2$
c. $[T]_B = \begin{bmatrix} 28 & 27 \\ 27 & 28 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{1}{55}\begin{bmatrix} 28 & -27 \\ -27 & 28 \end{bmatrix}$ e. $-4\sinh(3x) + 5\cosh(3x)$.

31. a. $W = Span(B),\ B = \{x\sin(2x),\ x\cos(2x),\ \sin(2x),\ \cos(2x)\}$ b. $T = 6I_W + 4D + 3D^2$
c. $[T]_B = \begin{bmatrix} -6 & -8 & 0 & 0 \\ 8 & -6 & 0 & 0 \\ 4 & -12 & -6 & -8 \\ 12 & 4 & 8 & -6 \end{bmatrix}$ d. $[T]_B^{-1} = \frac{1}{1250}\begin{bmatrix} -75 & 100 & 0 & 0 \\ -100 & -75 & 0 & 0 \\ 158 & 6 & -75 & 100 \\ -6 & 158 & -100 & -75 \end{bmatrix}$
e. $-7x\sin(2x) + 5x\cos(2x) + 4\sin(2x) - 6\cos(2x)$

32. d. False.

33. $-8 + 3x - 4x^2 + x^3/2$

34. $-3 + 19x - \frac{3}{2}x^2$

35. $13 - 11x + 8x^2$

36. $9 + 5x - \frac{3}{2}x^2 + 2x^3$

37. $3 + 4x - 7x^3$

38. $9 - 3x - 8x^2 + 5x^3 - 7x^4$

39. a. It is a diagonal matrix where none of the diagonal entries is 0. b. $\langle -147, 559/2, -632 \rangle$
c. $[T^{-1}]_{B',B} = Diag(1/3, 2, -1/5)$ d. $-\frac{92}{3} + \frac{74}{5}x + \frac{2}{5}x^2$

40. a. It is a triangular matrix where none of the diagonal entries is 0. b.
$\langle -26, 109/3, -175/3 \rangle$
c. $[T^{-1}]_{B',B} = \begin{bmatrix} -\frac{1}{2} & -\frac{15}{2} & -\frac{37}{2} \\ 0 & 3 & 6 \\ 0 & 0 & -1 \end{bmatrix}$ d. $79 + 44x + 2x^2$

41. a. $\frac{1}{20}\begin{bmatrix} -5 & 5 & 5 \\ 8 & 4 & -4 \\ 1 & 3 & 7 \end{bmatrix}$ b. $34 - 29x + 7x^2$.

Chapter Four Exercises

4.1 Exercises

1. a. $\{\langle 1,-1,-12,6\rangle,\langle 11,-16,13,1\rangle,\langle 1,1,-16,10\rangle\}$; $dim(V\vee W)=3$.
   b. $\{\langle 5,-7,-2,4\rangle\}$; $dim(V\cap W)=1$, c.
   $\langle 5,-7,-2,4\rangle=\frac{3}{5}\langle 1,-1,-12,6\rangle+\frac{2}{5}\langle 11,-16,13,1\rangle$, and
   $\langle 5,-7,-2,4\rangle=\frac{1}{3}\langle 1,1,-16,10\rangle+\frac{2}{3}\langle 7,-11,5,1\rangle$. d. $3=2+2-1$.
2. a. $\{\langle 3,5,-2,4\rangle,\langle 1,2,7,-3\rangle,\langle 0,2,1,-5\rangle,\langle 2,-3,1,6\rangle\}$; $dim(V\vee W)=4$ i.e. $V\vee W=\mathbb{R}^4$.
   b. $dim(V\cap W)=0$, so it has no basis. $4=2+2-0$, verifying d..
3. a. $\{\langle -3,-2,7,-4\rangle,\langle -2,13,-12,-2\rangle,\langle -2,3,-5,1\rangle,\langle -3,-5,6,-11\rangle\}$; $dim(V\vee W)=4$ i.e. $V\vee W=\mathbb{R}^4$.
   b. $\{\langle -26,-17,14,0\rangle,\langle 3,-8,0,7\rangle\}$; $dim(V\cap W)=2$
   c. $\langle -26,-17,14,0\rangle=4\langle -3,-2,7,-4\rangle-3\langle -2,13,-12,-2\rangle+10\langle -2,3,-5,1\rangle$, and
   $\langle 3,-8,0,7\rangle=-\langle -3,-2,7,-4\rangle-\langle -2,13,-12,-2\rangle+\langle -2,3,-5,1\rangle$;
   $\langle -26,-17,14,0\rangle=4\langle -3,-5,6,-11\rangle-3\langle -1,16,-8,8\rangle-17\langle 1,-3,2,-4\rangle$, and
   $\langle 3,-8,0,7\rangle=-\langle -3,-5,6,-11\rangle-\langle -1,16,-8,8\rangle-\langle 1,-3,2,-4\rangle$. d. $4=3+3-2$.
4. a. $\{\langle -3,4,-1,4,6\rangle,\langle -6,8,5,15,-13\rangle,\langle 1,-2,0,-5,3\rangle,\langle 1,3,-2,7,2\rangle\}$; $dim(V\vee W)=4$.
   b. $\{\langle 3,-2,-5,0,4\rangle\}$; $dim(V\cap W)=1$.
   c. $\langle 3,-2,-5,0,4\rangle=0\langle -3,4,-1,4,6\rangle-\langle -6,8,5,15,-13\rangle-3\langle 1,-2,0,-5,3\rangle$, and
   $\langle 3,-2,-5,0,4\rangle=-\langle 1,3,-2,7,2\rangle-\langle -4,-1,7,-7,-6\rangle$. d. $4=3+2-1$.
5. a. $\{\langle -1,7,5,-6,6\rangle,\langle -1,-8,2,-4,2\rangle,\langle 1,0,3,-4,3\rangle,\langle 5,3,-2,7,-4\rangle,\langle -6,9,-2,0,0\rangle\}$;
   $dim(V\vee W)=5$ i.e. $V\vee W=\mathbb{R}^5$. b. $\{\langle -17,31,-3,0,4\rangle,\langle -3,7,-1,2,0\rangle\}$;
   $dim(V\cap W)=2$.
   c.
   $\langle -17,31,-3,0,4\rangle=3\langle -1,7,5,-6,6\rangle-2\langle -1,-8,2,-4,2\rangle-6\langle 1,0,3,-4,3\rangle-2\langle 5,3,-2,7,-4\rangle$
   and
   $\langle -3,7,-1,2,0\rangle=\langle -1,7,5,-6,6\rangle-2\langle 1,0,3,-4,3\rangle$;
   $\langle -17,31,-3,0,4\rangle=3\langle -6,9,-2,0,0\rangle-2\langle -5,1,-3,-3,-2\rangle+3\langle -3,2,-1,-2,0\rangle$, and
   $\langle -3,7,-1,2,0\rangle=\langle -6,9,-2,0,0\rangle-\langle -3,2,-1,-2,0\rangle$. d. $5=4+3-2$.
6. a. $\{6-x+2x^2+10x^3,11-3x+6x^2+2x^3,3+17x+5x^2+4x^3\}$; $dim(V\vee W)=3$
   b. $\{-3+x-2x^2+2x^3\}$; $dim(V\cap W)=1$.
   c. $-3+x-2x^2+2x^3=\frac{2}{7}(6-x+2x^2+10x^3)-\frac{3}{7}(11-3x+6x^2+2x^3)$, and
   $-3+x-2x^2+2x^3=2(3+17x+5x^2+4x^3)-3(3+11x+4x^2+2x^3)$ d.
   $3=2+2-1$.
7. a. $\{2+5x-10x^2+5x^3,6-7x-4x^2+x^3,-8+14x-16x^2+3x^3\}$; $dim(V\vee W)=4$,
   i.e. $V\vee W=\mathbb{P}^3$. b. $\{4-x-7x^2+3x^3\}$; $dim(V\cap W)=1$.
   c. $4-x-7x^2+3x^3=\frac{1}{2}(2+5x-10x^2+5x^3)+\frac{1}{2}(6-7x-4x^2+x^3)$, and
   $4-x-7x^2+3x^3=\frac{3}{5}(2+3x-19x^2+13x^3)+\frac{2}{5}(7-7x+11x^2-12x^3)$ d.
   $4=3+2-1$.
8. a. $\{-3-2x+4x^2+x^4,6-3x^2+5x^3-5x^4,-7-7x+8x^2+2x^3+8x^4,$
   $-5-2x+7x^2+x^3-x^4,1-6x+3x^2-2x^3-4x^4\}$; $dim(V\vee W)=5$, i.e. $V\vee W=\mathbb{P}^4$.
   b. $\{5008+9057x-12636x^2,28-33x+52x^3,18+15x-52x^4\}$; $dim(V\cap W)=3$.
   c. $56-x-52x^2=30(-3-2x+4x^2+x^4)+5(6-3x^2+5x^3-5x^4)-$
   $3(-7-7x+8x^2+2x^3+8x^4)-19(-5-2x+7x^2+x^3-x^4)$;

$28 - 33x + 52x^3 = 2(-3 - 2x + 4x^2 + x^4) + 9(6 - 3x^2 + 5x^3 - 5x^4) +$
$5(-7 - 7x + 8x^2 + 2x^3 + 8x^4) - 3(-5 - 2x + 7x^2 + x^3 - x^4)$, and
$18 + 15x - 52x^4 = 18(-3 - 2x + 4x^2 + x^4) + 3(6 - 3x^2 + 5x^3 - 5x^4) -$
$7(-7 - 7x + 8x^2 + 2x^3 + 8x^4) - (-5 - 2x + 7x^2 + x^3 - x^4)$;
$56 - x - 52x^2 = -26(1 - 6x + 3x^2 - 2x^3 - 4x^4) + 5(5 - 14x + 7x^2 + x^3 - 12x^4) -$
$3(-9x + 3x^2 + 2x^4) - 19(-3 + 6x + 3x^3 + 2x^4)$;
$28 - 33x + 52x^3 = -26(1 - 6x + 3x^2 - 2x^3 - 4x^4) + 9(5 - 14x + 7x^2 + x^3 - 12x^4) +$
$5(-9x + 3x^2 + 2x^4) - 3(-3 + 6x + 3x^3 + 2x^4)$;

$18 + 15x - 52x^4 = 3(5 - 14x + 7x^2 + x^3 - 12x^4) - 7(-9x + 3x^2 + 2x^4) - (-3 + 6x + 3x^3 + 2x$
d. $5 = 4 + 4 - 3$.

11. $6 \le dim(V \cap W) \le 8$. 13. W must be a subspace of V. 14. W must be a subspace of V.
15. $V \cap W = W$, or $V \vee W = V$.

4.2 Exercises

1. a. $\{\langle 1,0,4\rangle, \langle 0,1,-7\rangle\}$ b. $\{\langle 3,5,4,-1\rangle, \langle 2,3,2,-1\rangle\}$ c. $\begin{bmatrix} 17 & -28 \\ -28 & 50 \end{bmatrix}$ d. $\begin{bmatrix} \frac{25}{33} & \frac{14}{33} \\ \frac{14}{33} & \frac{17}{66} \end{bmatrix}$
2. a. $\{\langle 1,-3,0\rangle, \langle 0,0,1\rangle\}$ b. $\{\langle 2,-3,-4,5\rangle, \langle -7,-1,9,3\rangle\}$ c. $\begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} \frac{1}{10} & 0 \\ 0 & 1 \end{bmatrix}$
3. a. $\{\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle\}$ b. $\{\langle 2,-3,-4,5\rangle, \langle -6,9,12,-5,\rangle, \langle -7,-1,9,3\rangle\}$ c. I_3, with inverse d. I_3. Note, though that this is not the identity transformation.
4. a. $\{\langle 1,0,-2,-2\rangle, \langle 0,1,2,1\rangle\}$ b. $\{\langle 3,2,-2\rangle, \langle 5,3,-1\rangle\}$ c. $\begin{bmatrix} 9 & -6 \\ -6 & 6 \end{bmatrix}$ d. $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}$
5. a. $\{\langle 1,0,-2\rangle, \langle 0,1,1\rangle\}$ b. $\{\langle 3,2,-2\rangle, \langle 5,3,-1\rangle\}$ c. $\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$ d. $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{6} \end{bmatrix}$
6. a. $\{\langle 1,5,0,4\rangle, \langle 0,0,1,-3\rangle\}$ b. $\{\langle 2,3,-4\rangle, \langle 5,7,-9\rangle\}$ c. $\begin{bmatrix} 42 & -12 \\ -12 & 10 \end{bmatrix}$ d. $\begin{bmatrix} \frac{5}{138} & \frac{1}{23} \\ \frac{1}{23} & \frac{7}{46} \end{bmatrix}$
7. a. $\{\langle 1,5,0,0\rangle, \langle 0,0,1,0\rangle, \langle 0,0,0,1\rangle\}$ b. $\{\langle 2,3,-4\rangle, \langle 5,7,-9\rangle, \langle -7,-9,8\rangle\}$
   c. $\begin{bmatrix} 26 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} \frac{1}{26} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
8. a. $\{\langle 1,4,0,-5,2\rangle, \langle 0,0,1,6,-3\rangle\}$ b. $\{\langle -5,3,2,-4\rangle, \langle -4,-2,3,1\rangle\}$
   c. $\begin{bmatrix} 46 & -36 \\ -36 & 46 \end{bmatrix}$ d. $\begin{bmatrix} \frac{23}{410} & \frac{9}{205} \\ \frac{9}{205} & \frac{23}{410} \end{bmatrix}$
9. a. $\{\langle 1,4,0,-5\rangle, \langle 0,0,1,6\rangle\}$ b. $\{\langle -5,3,2,-4\rangle, \langle -4,-2,3,1\rangle\}$

c. $\begin{bmatrix} 42 & -30 \\ -30 & 37 \end{bmatrix}$ d. $\begin{bmatrix} \frac{37}{654} & \frac{5}{109} \\ \frac{5}{109} & \frac{7}{109} \end{bmatrix}$

10. a. $\{\langle 1,-2,6,0,-4\rangle, \langle 0,0,0,1,5\rangle\}$ b. $\{\langle -5,2,-3,4\rangle, \langle -3,-3,2,-5\rangle\}$
c. $\begin{bmatrix} 57 & -20 \\ -20 & 26 \end{bmatrix}$ d. $\begin{bmatrix} \frac{13}{541} & \frac{10}{541} \\ \frac{10}{541} & \frac{57}{1082} \end{bmatrix}$

11. a. $\{\langle 1,-3,0,0,2\rangle, \langle 0,0,1,0,-4\rangle, \langle 0,0,0,1,7\rangle\}$ b. $\{\langle -2,3,4,-5\rangle, \langle -3,7,-1,2\rangle, \langle -5,4,-2,3\rangle\}$
c. $\begin{bmatrix} 14 & -8 & 14 \\ -8 & 17 & -28 \\ 14 & -28 & 50 \end{bmatrix}$ d. $\begin{bmatrix} \frac{33}{332} & \frac{1}{83} & -\frac{7}{332} \\ \frac{1}{83} & \frac{63}{83} & \frac{35}{83} \\ -\frac{7}{332} & \frac{35}{83} & \frac{87}{332} \end{bmatrix}$

12. a. $\{\langle 1,-3,0,0,0\rangle, \langle 0,0,1,0,0\rangle, \langle 0,0,0,1,0\rangle, \langle 0,0,0,0,1\rangle\}$
b. $\{\langle -2,3,4,-5\rangle, \langle -3,7,-1,2\rangle, \langle -5,4,-2,3\rangle, \langle -7,6,-2,3\rangle\}$
c. $\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} \frac{1}{10} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

13. a. $\{\langle 1,5,0,-2\rangle, \langle 0,0,1,6\rangle\}$ b. $\{\langle 2,-4,3,5,-6\rangle, \langle -1,1,1,-3,2\rangle\}$
c. $\begin{bmatrix} 30 & -12 \\ -12 & 37 \end{bmatrix}$ d. $\begin{bmatrix} \frac{37}{966} & \frac{2}{161} \\ \frac{2}{161} & \frac{5}{161} \end{bmatrix}$

14. a. $\{\langle 1,5,0,0\rangle, \langle 0,0,1,0\rangle, \langle 0,0,0,1\rangle\}$ b. $\{\langle 2,-4,3,5,-6\rangle, \langle -1,1,1,-3,2\rangle, \langle -9,14,0,-28,24\rangle\}$
c. $\begin{bmatrix} 26 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} \frac{1}{26} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

15. a. $\{\langle 1,0,0,-2\rangle, \langle 0,1,0,3\rangle, \langle 0,0,1,-5\rangle\}$ b. $\{\langle -2,5,1,-2,-1\rangle, \langle 1,-1,1,-2,1\rangle, \langle 1,-1,-1,1,1\rangle\}$
c. $\begin{bmatrix} 5 & -6 & 10 \\ -6 & 10 & -15 \\ 10 & -15 & 26 \end{bmatrix}$ d. $\begin{bmatrix} \frac{35}{39} & \frac{2}{13} & -\frac{10}{39} \\ \frac{2}{13} & \frac{10}{13} & \frac{5}{13} \\ -\frac{10}{39} & \frac{5}{13} & \frac{14}{39} \end{bmatrix}$

16. a. The standard basis for $\mathbb{R}^4$ b. The four columns of $[T]$. c. I_3 d. I_3. Again, this is ***not*** the identity transformation.

4.3 Exercises

1. a. $\{\langle -5,1,12,11\rangle\}$ b. $\{\langle 1,1,0\rangle, \langle -4,7,1\rangle\}$
2. a. $\{\langle -7,-1,9,3\rangle, \langle 2,-3,-4,5\rangle\}$ (scaled down) b. $\{\langle 2,0,1\rangle, \langle 3,1,0\rangle, \langle 0,0,1\rangle\}$
3. a. $\{\langle -7,-1,9,7\rangle, \langle 2,-3,-4,3\rangle\}$ (scaled down) b. $\{\langle 1,0,1\rangle, \langle 3,1,0\rangle\}$ (scaled up)
4. a. $\{\langle 3,2,-2\rangle, \langle 5,2,6\rangle\}$ (scaled down) b. $\{\langle -1,1,0,0\rangle, \langle 2,1,0,0\rangle, \langle 2,-2,1,0\rangle, \langle 2,-1,0,1\rangle\}$

5. a. $\{\langle 5,4,-8\rangle,\langle 3,2,-2\rangle\}$ b. $\{\langle -1,1,0\rangle,\langle 2,-1,1\rangle,\langle 1,-1,1\rangle\}$
6. a. $\{\langle 4,7,-10\rangle,\langle 13,17,-21\rangle\}$ b. $\{\langle -1,0,1,0\rangle,\langle -5,1,0,0\rangle,\langle -4,0,3,1\rangle\}$
7. a. $\{\langle 8,10,-9\rangle,\langle -9,-14,16\rangle,\langle 23,38,-65\rangle\}$ or simply $\left\{\vec{i},\vec{j},\vec{k}\right\}$ b. $\{\langle -1,0,1,0\rangle,\langle -15,0,5,-1\rangle,\langle -5,1,0,0\rangle\}$
8. a. $\{\langle -43,105,-8,-110\rangle,\langle -9,1,5,-3\rangle\}$ (scaled down) b. $\{\langle -1,0,1,0,0\rangle,\langle -4,1,0,0\rangle,\langle 5,0,-6,1,0\rangle,\langle -2,0,3,0,1\rangle\}$
9. a. $\{\langle -88,22,45,-41\rangle,\langle -92,350,-57,-355\rangle\}$ b. $\{\langle 1,0,1,0\rangle,\langle -4,1,0,0\rangle,\langle 5,0,-6,1\rangle\}$
10. a. $\{\langle 192,3,43,-13\rangle\}$ (scaled down) b. $\{\langle 1,0,0,-1,0\rangle,\langle 2,1,0,0,0\rangle,\langle -6,0,1,0,0\rangle,\langle 4,0,0,-5,1\rangle\}$
11. a. $\{\langle 15,13,45,-58\rangle\}$ b. $\{\langle 3,1,0,0,0\rangle,\langle -2,0,4,-7,1\rangle\}$
12. a. $\{\langle -88,97,7,-1\rangle,\langle 4,-40,-79,100\rangle\}$
b. $\left\{\left\langle -\frac{1}{10},0,\frac{1}{5},-\frac{7}{20},\frac{1}{20}\right\rangle,\left\langle -\frac{1}{10},0\frac{1}{5},-\frac{17}{20},\frac{11}{20}\right\rangle,\left\langle \frac{9}{10},0,\frac{1}{5},\frac{143}{20},-\frac{109}{20}\right\rangle,\langle 3,1,0,0,0\rangle\right\}$
13. a. $\{\langle 12,-22,13,31,-34\rangle,\langle -61,85,1,-171,146\rangle\}$ b. $\{\langle 2,0,1,0\rangle,\langle -5,1,0,0\rangle,\langle 2,0,-6,1\rangle\}$
14. a. $\{\langle -21,31,3,-65,54\rangle,\langle -53,78,8,-164,136\rangle\}$ b. $\left\{\langle 2,0,-6,1\rangle,\langle -\frac{3}{4},0,\frac{3}{4},-\frac{1}{4}\rangle,\left\langle -\frac{1}{8},0,-\frac{11}{8},-\frac{1}{8}\right\rangle,\langle -5,1,0,0\rangle\right\}$. Since this basis has 4 elements, the preimage is all of $\mathbb{R}^4$, so any basis for $\mathbb{R}^4$ is also a correct answer, including the standard basis.
15. a. $\{\langle -6,21,-11,12,-1\rangle,\langle 7,-25,19,-23,1\rangle,\langle -2,14,-12,13,2\rangle\}$ b. $\{\langle 0,2,-1,0,0\rangle,\langle 2,-3,5,1\rangle\}$

4.4 Exercises

1. Yes. 2. No. 3. Yes. 4. No. 5. Yes. 6. No. 7. Yes. 8. Yes. 9. No. 10. Yes.
11. $x_0 = -21$ and $z_0 = 14$. 12. $x_0 = 30$, $y_0 = -45$, and $z_0 = 18$.
13. a. $\{\langle 3,-1,2,0\rangle,\vec{e}_1,\vec{e}_2,\vec{e}_4\}$ b. $\{\vec{e}_1,\vec{e}_2,\vec{e}_4\}$ c. 3; d. $3 = 4-1$
14. a. $\{\langle 3,5,2,-2\rangle,\langle -2,1,2,-2\rangle,\vec{e}_1,\vec{e}_3\}$ b. $\{\vec{e}_1,\vec{e}_3\}$ c. 2; d. $2 = 4-2$
15. a. $\{\langle 3,0,-2,0,7\rangle,\vec{e}_1,\vec{e}_2,\vec{e}_3,\vec{e}_4\}$ b. $\{\vec{e}_1,\vec{e}_2,\vec{e}_3,\vec{e}_4\}$ c. 4; d. $4 = 5-1$
16. a. $\{\langle 2,0,7,3,0\rangle,\langle 0,5,-14,-6,0\rangle,\vec{e}_1,\vec{e}_3,\vec{e}_5\}$ b. $\{\vec{e}_1,\vec{e}_3,\vec{e}_5\}$ c. 3; d. $3 = 5-2$
17. a. $\{\langle 4,-3,0,0,5\rangle,\langle 2,-3,0,0,5\rangle,\langle 2,1,0,0,5\rangle,\vec{e}_3,\vec{e}_4\}$ b. $\{\vec{e}_3,\vec{e}_4\}$ c. 2 d. $2 = 5-3$

4.5 Exercises

1. a. $\{\langle 3,5,4,-1\rangle,\langle 2,3,2,-1\rangle\}$ b. $\{\langle -4,7,1\rangle\}$ c. $\{\vec{e}_1+ker(T),\vec{e}_2+ker(T)\}$
d. $\widetilde{T}(\vec{e}_1+ker(T)) = \vec{c}_1; \widetilde{T}(\vec{e}_2+ker(T)) = \vec{c}_2$
2. a. $\{\langle 2,-3,-4,5\rangle,\langle -7,-1,9,3\rangle\}$ b. $\{\langle 3,1,0\rangle\}$ c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T)\}$
d. $\widetilde{T}(\vec{e}_1+ker(T)) = \vec{c}_1; \widetilde{T}(\vec{e}_3+ker(T)) = \vec{c}_3$
3. a. $\{\langle 3,2,-2\rangle,\langle 5,3,-1\rangle\}$ b. $\{\langle 2,-2,1,0\rangle,\langle 2,-1,0,1\rangle\}$ c. $\{\vec{e}_1+ker(T),\vec{e}_2+ker(T)\}$
d. $\widetilde{T}(\vec{e}_1+ker(T)) = \vec{c}_1; \widetilde{T}(\vec{e}_2+ker(T)) = \vec{c}_2$
4. a. $\{\langle 3,2,-2\rangle,\langle 5,3,-1\rangle\}$ b. $\{\langle 2,1,0\rangle\}$ c. $\{\vec{e}_1+ker(T),\vec{e}_2+ker(T)\}$ d. $\widetilde{T}(\vec{e}_1+ker(T)) = \vec{c}_1; \widetilde{T}(\vec{e}_2+ker(T)) = \vec{c}_2$
5. a. $\{\langle 2,3,-4\rangle,\langle 5,7,-9\rangle\}$ b. $\{\langle -5,1,0,0\rangle,\langle -4,3,0,1\rangle\}$ c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T)\}$
d. $\widetilde{T}(\vec{e}_1+ker(T)) = \vec{c}_1; \widetilde{T}(\vec{e}_3+ker(T)) = \vec{c}_3$
6. a. $\{\langle 2,3,-4\rangle,\langle 5,7,-9\rangle,\langle -7,-9,8\rangle\}$ b. $\{\langle -5,1,0,0\rangle\}$ c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T),\vec{e}_4+ker(T)\}$
d. $\widetilde{T}(\vec{e}_1+ker(T)) = \vec{c}_1; \widetilde{T}(\vec{e}_3+ker(T)) = \vec{c}_3; \widetilde{T}(\vec{e}_4+ker(T)) = \vec{c}_4$
7. a. $\{\langle -5,3,2,-4\rangle,\langle -4,-2,3,1\rangle\}$ b. $\{\langle -4,1,0,0,0\rangle,\langle 5,0,-6,1,0\rangle,\langle -2,0,3,0,1\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T)\}$ d. $\widetilde{T}(\vec{e}_1+ker(T)) = \vec{c}_1; \widetilde{T}(\vec{e}_3+ker(T)) = \vec{c}_3$

8. a. $\{\langle -5,3,2,-4\rangle,\langle -4,-2,3,1\rangle\}$ b. $\{\langle -4,1,0,0,0\rangle,\langle 5,0,-6,1,0\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T)\}$ d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3$

9. a. $\{\langle -5,2,-3,4\rangle,\langle -3,-3,2,5\rangle\}$ b. $\{\langle 2,1,0,0,0\rangle,\langle -6,0,1,0,0\rangle,\langle 4,0,0,-5,1\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_4+ker(T)\}$ d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_4+ker(T))=\vec{c}_4$

10. a. $\{\langle -2,3,4,-5\rangle,\langle -3,7,-1,2\rangle,\langle -5,4,-2,3\rangle\}$ b. $\{\langle 3,1,0,0,0\rangle,\langle -2,0,4,-7,1\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T),\vec{e}_4+ker(T)\}$ d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3;$
$\widetilde{T}(\vec{e}_4+ker(T))=\vec{c}_4$

11. a. $\{\langle -2,3,4,-5\rangle,\langle -3,7,-1,2\rangle,\langle -5,4,-2,3\rangle,\langle -7,6,-2,3\rangle\}$ b. $\{\langle 3,1,0,0,0\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T),\vec{e}_4+ker(T),\vec{e}_5+ker(T)\}$
d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3;\widetilde{T}(\vec{e}_4+ker(T))=\vec{c}_4;\widetilde{T}(\vec{e}_5+ker(T))=\vec{c}_5$

12. a. $\{\langle 2,-4,3,5,-6\rangle,\langle -1,1,1,-3,2\rangle\}$ b. $\{\langle -5,1,0,0\rangle,\langle 2,0,-6,1\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T)\}$ d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3$

13. a. $\{\langle 2,-4,3,5,-6\rangle,\langle -1,1,1,-3,2\rangle,\langle -9,14,0,-28,24\rangle\}$ b. $\{\langle -5,1,0,0\rangle\}$ c.
$\{\vec{e}_1+ker(T),\vec{e}_3+ker(T),\vec{e}_4+ker(T)\}$
d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3;\widetilde{T}(\vec{e}_4+ker(T))=\vec{c}_4$

14. a. $\{\langle -2,5,1,-2,-1\rangle,\langle 1,-1,1,-2,1\rangle,\langle 1,-1,-1,1,1\rangle\}$ b. $\{\langle 2,-3,5,1\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_2+ker(T),\vec{e}_3+ker(T)\}$ d.
$\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_2+ker(T))=\vec{c}_2;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3$

15. a. $\{\langle\langle -2,-1,1\rangle\rangle+U\}$ b. $\{\vec{e}_2+W\}$ c. $\{\langle -2,-1,1\rangle+U,\vec{e}_2+U\}$ d. $\{\vec{e}_2+W/U\}$ e.
$\widetilde{T}(\vec{e}_2+U+W/U)=\vec{e}_2+W$

16. a. $\{\langle 1,1,1,2\rangle+U\}$ b. $\{\vec{e}_1+W,\vec{e}_3+W\}$ c. $\{\langle 1,1,1,2\rangle+U,\vec{e}_1+U,\vec{e}_3+U\}$
d. $\{\vec{e}_1+W/U,\vec{e}_3+W/U\}$ e. $\widetilde{T}(\vec{e}_1+U+W/U)=\vec{e}_1+W;\widetilde{T}(\vec{e}_3+U+W/U)=\vec{e}_3+W;$

17. a. $\{\langle 1,1,1,2\rangle+U,\langle 3,-1,1,2\rangle+U\}$ b. $\{\vec{e}_3+W\}$ c.
$\{\langle 1,1,1,2\rangle+U,\langle 3,-1,1,2\rangle+U,\vec{e}_3+U\}$
d. $\{\vec{e}_3+W/U\}$ e. $\widetilde{T}(\vec{e}_3+U+W/U)=\vec{e}_3+W$

18. a. $\{\langle 3,-1,1,2\rangle+U\}$ b. $\{\vec{e}_3+W\}$ c. $\{\langle 3,-1,1,2\rangle+U,\vec{e}_3+U\}$ d. $\{\vec{e}_3+W/U\}$ e.
$\widetilde{T}(\vec{e}_3+U+W/U)=\vec{e}_3+W$

19. a. $\{\langle 1,1,1,2,-3\rangle+U,\langle 3,-1,1,2,-3\rangle+U\}$ b. $\{\vec{e}_3+W,\vec{e}_4+W\}$
c. $\{\langle 1,1,1,2,-3\rangle+U,\langle 3,-1,1,2,-3\rangle+U,\vec{e}_3+U,\vec{e}_4+U\}$
d. $\{\vec{e}_3+W/U,\vec{e}_4+W/U\}$ e. $\widetilde{T}(\vec{e}_3+U+W/U)=\vec{e}_3+W;\widetilde{T}(\vec{e}_4+U+W/U)=\vec{e}_4+W$

20. a. $\{\langle 3,-1,1,2,-3\rangle+U\}$ b. $\{\vec{e}_3+W,\vec{e}_4+W\}$ c. $\{\langle 3,-1,1,2,-3\rangle+U,\vec{e}_3+U,\vec{e}_4+U\}$
d. $\{\vec{e}_3+W/U,\vec{e}_4+W/U\}$ e. $\widetilde{T}(\vec{e}_3+U+W/U)=\vec{e}_3+W;\widetilde{T}(\vec{e}_4+U+W/U)=\vec{e}_4+W$

21. a. $\{\langle 3,-1,1,2,-3\rangle+U,\langle 3,-1,1,-1,-3\rangle+U\}$ b. $\{\vec{e}_3+W\}$
c. $\{\langle 3,-1,1,2,-3\rangle+U,\langle 3,-1,1,-1,-3\rangle+U,\vec{e}_3+U\}$ d. $\{\vec{e}_3+W/U\}$ e.
$\widetilde{T}(\vec{e}_3+U+W/U)=\vec{e}_3+W$

22. a. $\{\langle 1,-1,-12,6\rangle,\langle 11,-16,13,1\rangle,\langle 1,1,-16,10\rangle\}$ b. $\{\langle 5,-7,-2,4\rangle\}$
c. $\{\langle 1,-1,-12,6\rangle+W,\langle 1,1,-16,10\rangle+W\}$ d.
$\{\langle 1,-1,-12,6\rangle+(V\cap W),\langle 1,1,-16,10\rangle+(V\cap W)\}$
e. $\{\langle 1,-1,-12,6\rangle+V,\langle 1,1,-16,10\rangle+V\}$ f.
$\{\langle 1,-1,-12,6\rangle+(V\cap W),\langle 1,1,-16,10\rangle+(V\cap W)\}$
g. $\widetilde{T}_1(\langle 1,-1,-12,6\rangle+W)=\langle 1,-1,-12,6\rangle+(V\cap W);$
$\widetilde{T}_1(\langle 1,1,-16,10\rangle+W)=\langle 1,1,-16,10\rangle+(V\cap W);$
h. $\widetilde{T}_2(\langle 1,-1,-12,6\rangle+V)=\langle 1,-1,-12,6\rangle+(V\cap W);$
$\widetilde{T}_2(\langle 1,1,-16,10\rangle+V)=\langle 1,1,-16,10\rangle+(V\cap W)$

23. a. $\{\langle 3,5,-2,4\rangle,\langle 1,2,7,-3\rangle,\langle 0,2,1,-5\rangle,\langle 2,-3,1,6\rangle\}$ b. $dim(V\cap W)=0,$ so it has no

basis.

c. $\{\langle 3,5,-2,4\rangle + W, \langle 1,2,7,-3\rangle + W\}$ d. $\left\{\langle 3,5,-2,4\rangle + \left\{\vec{0}_4\right\}, \langle 1,2,7,-3\rangle + \left\{\vec{0}_4\right\}\right\}$

e. $\{\langle 0,2,1,-5\rangle + V, \langle 2,-3,1,6\rangle + V\}$ f. $\left\{\langle 0,2,1,-5\rangle + \left\{\vec{0}_4\right\}, \langle 2,-3,1,6\rangle + \left\{\vec{0}_4\right\}\right\}$

g.
$\widetilde{T}_1(\langle 3,5,-2,4\rangle + W) = \langle 3,5,-2,4\rangle + \left\{\vec{0}_4\right\}; \widetilde{T}_1(\langle 1,2,7,-3\rangle + W) = \langle 1,2,7,-3\rangle + \left\{\vec{0}_4\right\};$

h. $\widetilde{T}_2(\langle 0,2,1,-5\rangle + V) = \langle 0,2,1,-5\rangle + \left\{\vec{0}_4\right\}; \widetilde{T}_2(\langle 2,-3,1,6\rangle + V) = \langle 2,-3,1,6\rangle + \left\{\vec{0}_4\right\}$

24. a. $\{\langle -3,-2,7,-4\rangle, \langle -2,13,-12,-2\rangle, \langle -2,3,-5,1\rangle, \langle -3,-5,6,-11\rangle\}$
b. $\{\langle -26,-17,14,0\rangle, \langle 3,-8,0,7\rangle\}$ c. $\{\langle -3,-2,7,-4\rangle + W\}$ d. $\{\langle -3,-2,7,-4\rangle + (V \cap W)\}$
e. $\{\langle -3,-5,6,-11\rangle + V\}$ f. $\{\langle -3,-5,6,-11\rangle + (V \cap W)\}$
g. $\widetilde{T}_1(\langle -3,-2,7,-4\rangle + W) = \langle -3,-2,7,-4\rangle + (V \cap W)$ h.
$\widetilde{T}_2(\langle -3,-5,6,-11\rangle + V) = \langle -3,-5,6,-11\rangle + (V \cap W)$

25. a. $\{\langle -3,4,-1,4,6\rangle, \langle -6,8,5,15,-13\rangle, \langle 1,-2,0,-5,3\rangle, \langle 1,3,-2,7,2\rangle\}$ b. $\{\langle 3,-2,-5,0,4\rangle\}$
c. $\{\langle -3,4,-1,4,6\rangle + W, \langle -6,8,5,15,-13\rangle + W\}$ d.
$\{\langle -3,4,-1,4,6\rangle + (V \cap W), \langle -6,8,5,15,-13\rangle + (V \cap W)\}$
e. $\{\langle 1,3,-2,7,2\rangle + V\}$ f. $\{\langle 1,3,-2,7,2\rangle + (V \cap W)\}$ g.
$\widetilde{T}_1(\langle -3,4,-1,4,6\rangle + W) = \langle -3,4,-1,4,6\rangle + (V \cap W);$
$\widetilde{T}_1(\langle -6,8,5,15,-13\rangle + W) = \langle -6,8,5,15,-13\rangle + (V \cap W);$ h.
$\widetilde{T}_2(\langle 1,3,-2,7,2\rangle + V) = \langle 1,3,-2,7,2\rangle + (V \cap W)$

26. a. $\{\langle -1,7,5,-6,6\rangle, \langle -1,-8,2,-4,2\rangle, \langle 1,0,3,-4,3\rangle, \langle 5,3,-2,7,-4\rangle, \langle -6,9,-2,0,0\rangle\}$
b. $\{\langle -17,31,-3,0,4\rangle, \langle -3,7,-1,2,0\rangle\}$ c. $\{\langle -1,7,5,-6,6\rangle + W, \langle -1,-8,2,-4,2\rangle + W\}$
d. $\{\langle -1,7,5,-6,6\rangle + (V \cap W), \langle -1,-8,2,-4,2\rangle + (V \cap W)\}$ e. $\{\langle -6,9,-2,0,0\rangle + V\}$ f.
$\{\langle -6,9,-2,0,0\rangle + (V \cap W)\}$
g.
$\widetilde{T}_1(\langle -1,7,5,-6,6\rangle + W) = \langle -1,7,5,-6,6\rangle + (V \cap W);$ $\widetilde{T}_1(\langle -1,-8,2,-4,2\rangle + W) = \langle -1,-8,2$
h. $\widetilde{T}_2(\langle -6,9,-2,0,0\rangle + V) = \langle -6,9,-2,0,0\rangle + (V \cap W)$

Chapter Five Exercises

5.1 Exercises

1. 3; 2. -23; 3. $-11/3$; 4. $-5\sqrt{3}$; 5. $4\ln 2 + 7\ln 3$; 6. 1/2; 7. -47 8. 148; 9. 27/8;

10. $-29/3$; 11. 1800; 12. $-70\ln 2 - 49\ln 5$

13. a. ab; b. it is invertible if and only if both a and b are non-zero; c. $\dfrac{1}{ab}\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$.

14. a. $a^2 + b^2$; b. it is invertible if and only if either a or b is non-zero; c. $\dfrac{1}{a^2+b^2}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

15. a. $a^2 - b^2$; b. it is invertible if and only if $a \neq \pm b$; c. $\dfrac{1}{a^2-b^2}\begin{bmatrix} a & -b \\ -b & a \end{bmatrix}$.

16. a. $2ab$; b. it is invertible if and only if both a and b are non-zero; c. $\dfrac{1}{2ab}\begin{bmatrix} b & -a \\ b & a \end{bmatrix} = \begin{bmatrix} \frac{1}{2a} & -\frac{1}{2b} \\ \frac{1}{2a} & \frac{1}{2b} \end{bmatrix}$.

17. a. $b - a$; b. it is invertible if and only if $a \neq b$; c. $\dfrac{1}{b-a}\begin{bmatrix} b & -a \\ -1 & 1 \end{bmatrix}$.

18. a. $2e^a$; b. it is always invertible; c. $\dfrac{e^{-a}}{2}\begin{bmatrix} e^{-a} & -e^{-a} \\ e^{2a} & e^{2a} \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} e^{-2a} & -e^{-2a} \\ e^{a} & e^{a} \end{bmatrix}$.

19. a. 1; b. it is always invertible; c. $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$.

20. a. 1; b. it is always invertible; c. $\begin{bmatrix} \cosh(a) & -\sinh(a) \\ -\sinh(a) & \cosh(a) \end{bmatrix}$

21. a. $\sin(\theta + \phi)$; b. it is invertible if and only if $\theta + \phi \neq n\pi$, where n is an integer; c. $\dfrac{1}{\sin(\theta+\phi)}\begin{bmatrix} \sin(\phi) & \sin(\theta) \\ -\cos(\phi) & \cos(\theta) \end{bmatrix}$

22. $(2,4,1,3)$; both have 3 inversions. 23. $(5,3,2,4,1)$; both have 8 inversions. Notice that $\sigma = \sigma^{-1}$.

24. $(5,3,6,1,4,2)$; both have 10 inversions. 25. $(6,4,2,7,5,1,3)$; both have 14 inversions.

26. $(5,7,3,8,4,1,6,2)$; both have 18 inversions. 27. $(2,1,4,3)$; 2 inversions.

28. $(2,3,5,4,1)$; 5 inversions. 29. $(4,1,2,5,6,3)$; 5 inversions. 30. $(6,3,4,2,5,1,7)$; 11 inversions.

31. $(6,2,3,5,1,7,8,4)$; 11 inversions. 32. a. 0; b. 0; c. -1. 33. a. 0; b. 0; c. -1.

34. $(c-a)(c-b)(b-a)$ (other factorizations are possible, up to ± 1)
35. the permutation $\sigma = (n, n-1, \ldots, 3, 2, 1)$ will have $(n-1) + \cdots + 3 + 2 + 1 = (n-1)n/2$ inversions.

5.2 Exercises

1. (−) 2. (+) 3. (−) 4. (+) 5. (−) 6. (−) 7. missing 2; (+) 8. missing 4; (−) 9. missing 3; (+)
10. missing 5; (+) 11. missing 2 and 5; (−) 12. missing 7 and 4; (−) 13. 0; column 2 is all zeroes.
14. 0; the third row is 4 times the first 15. −30; the matrix is upper triangular
16. 7/5; the matrix is upper triangular; 17. −2640; the matrix is lower triangular
18. 60; the matrix is upper triangular; 19. 3780; the matrix is upper triangular
20. −1/4; the matrix is lower triangular. 21. −560 22. 360 23. 720 24. −7/2
25. a. −5; b. 5; c. −20 d. 1/3 26. a. 42; b. 20 c. 10800 d. 40
27. a. 9 b. 270 c. −252 d. 0 28. a. −70; b. 6 c. 480 d. −588
29. −321; 30. 93; 31. 2981 32. 403 33. 863; 34. −1779; 35. −182 36. −448
37. −439; 38. 9730; 39. −29700 40. 214295 41. a. $det(A) = 76$; $det(B) = 345$; $det(C) = 421$

5.3 Exercises

1. a. $det(A) = -34$ and $det(B) = 46$. b. $AB = \begin{bmatrix} 38 & 36 \\ 16 & -26 \end{bmatrix}$ and $det(AB) = -1564$.
   c. $-1564 = (-34)(46)$ d. $A + B = \begin{bmatrix} 11 & 4 \\ 4 & 5 \end{bmatrix}$ and $det(A+B) = 39$.
   e. $39 \neq -34 + 46$. f. $3B = \begin{bmatrix} 18 & -12 \\ 3 & 21 \end{bmatrix}$ and $det(3B) = 414$. g. $det(3B) = 9det(B)$.
2. a. $7\begin{vmatrix} -1 & 3 & -4 \\ 2 & -8 & 3 \\ 6 & 5 & 7 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -3 & 2 \\ -1 & 3 & -4 \\ 2 & -8 & 3 \end{vmatrix}$ b. first determinant is −149 and the other is −27; c. −1097
3. a. $-6\begin{vmatrix} -3 & 7 & -2 \\ 3 & 6 & 4 \\ -8 & -2 & 3 \end{vmatrix} - (-4)\begin{vmatrix} 5 & -3 & -2 \\ -1 & 3 & 4 \\ 2 & -8 & 3 \end{vmatrix}$ b. first determinant is −449 and the other is 168; c. 3366
4. a. $\begin{bmatrix} 4 & -2 & 3 & 8 \\ 9 & 0 & 17 & 28 \\ 2 & 3 & -2 & 3 \\ -3 & 0 & 9 & -5 \end{bmatrix}$ b. $\begin{bmatrix} 6 & 1 & 1 & 11 \\ 9 & 0 & 17 & 28 \\ 2 & 3 & -2 & 3 \\ -3 & 0 & 9 & -5 \end{bmatrix}$

c. $\begin{bmatrix} 6 & 1 & 1 & 11 \\ 9 & 0 & 17 & 28 \\ -16 & 0 & -5 & -30 \\ -3 & 0 & 9 & -5 \end{bmatrix}$ d. $-\begin{vmatrix} 9 & 17 & 28 \\ -16 & -5 & -30 \\ -3 & 9 & -5 \end{vmatrix}$

e. 1627

5. a. –219 b. –180 6. a. –255 b. 2452 7. a. –511 b. –1578
8. a. 56 b. –43 9. a. –42 b. –686 14. 140
16. a. 1512 g. $r(x) = (x - a_1)(x - a_2)\cdots(x - a_k)$; the bottom entry will be: $r(a_{k+1}) = (a_{k+1} - a_1)(a_{k+1} - a_2)\cdots(a_{k+1} - a_k)$

5.4 Exercises

1. a. $adj(A) = \begin{bmatrix} 4 & -5 \\ -1 & 3 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} \frac{4}{7} & -\frac{5}{7} \\ -\frac{1}{7} & \frac{3}{7} \end{bmatrix}$. b. $adj(B) = \begin{bmatrix} 20 & 5 \\ -12 & -3 \end{bmatrix}$; B is not invertible.

2. a. $adj(A) = \begin{bmatrix} -4 & -7 & -2 \\ -10 & 5 & -5 \\ -1 & -13 & -23 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} \frac{4}{45} & \frac{7}{45} & \frac{2}{45} \\ \frac{2}{9} & -\frac{1}{9} & \frac{1}{9} \\ \frac{1}{45} & \frac{13}{45} & \frac{23}{45} \end{bmatrix}$. b.

$adj(B) = \begin{bmatrix} 14 & -6 & -31 \\ -7 & -24 & 2 \\ -35 & 15 & -17 \end{bmatrix}$; $B^{-1} = \begin{bmatrix} -\frac{2}{27} & \frac{2}{63} & \frac{31}{189} \\ \frac{1}{27} & \frac{8}{63} & -\frac{2}{189} \\ \frac{5}{27} & -\frac{5}{63} & \frac{17}{189} \end{bmatrix}$.

3. a. $adj(A) = \begin{bmatrix} 183 & 85 & 74 & -62 \\ -534 & -338 & -379 & -44 \\ -63 & -63 & 42 & -63 \\ -339 & -388 & -362 & 53 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -\frac{61}{343} & -\frac{85}{1029} & -\frac{74}{1029} & \frac{62}{1029} \\ \frac{178}{343} & \frac{338}{1029} & \frac{379}{1029} & \frac{44}{1029} \\ \frac{3}{49} & \frac{3}{49} & -\frac{2}{49} & \frac{3}{49} \\ \frac{113}{343} & \frac{388}{1029} & \frac{362}{1029} & -\frac{53}{1029} \end{bmatrix}$

4. a. $\langle x,y \rangle = \left\langle -\frac{31}{13}, -\frac{29}{13} \right\rangle$ b. $\langle x,y \rangle = \left\langle \frac{2}{59}, -\frac{92}{59} \right\rangle$
5. a. doesn't apply b. $\langle x,y \rangle = \left\langle \frac{3}{73}, -\frac{52}{73} \right\rangle$
6. a. $\langle x,y,z \rangle = \langle \frac{3}{4}, \frac{7}{4}, \frac{1}{2} \rangle$ b. doesn't apply
7. a. $\langle x,y,z \rangle = \left\langle \frac{209}{193}, \frac{66}{193}, \frac{367}{193} \right\rangle$ b. $\langle x,y,z \rangle = \left\langle -\frac{137}{83}, \frac{26}{83}, -\frac{15}{83} \right\rangle$
8. a. $\langle x,y,z,w \rangle = \left\langle \frac{164}{107}, -\frac{979}{107}, \frac{399}{107}, \frac{1029}{107} \right\rangle$ b. $\langle x,y,z,w \rangle = \langle 5,-8,6,0 \rangle$
9. a. $\langle x,y,z,w \rangle = \left\langle \frac{161}{44}, \frac{433}{88}, -\frac{247}{176}, -\frac{211}{44} \right\rangle$ b. $\langle x,y,z,w \rangle = \langle 1, -\frac{1}{2}, -\frac{1}{2}, 1 \rangle$

10. $b = \frac{3897}{6445}$; $d = -\frac{836}{6445}$ 11. $\langle 5, 0, -2, 7 \rangle$

12. $\begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$; $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$; $\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

16. b. $adj(A) = \begin{bmatrix} -21 & -35 & 27 \\ 0 & 14 & -12 \\ 0 & 0 & -6 \end{bmatrix}$. It is also upper triangular.

5.5 Exercises

1. a. $W_S(x) = 16\cos x \cos 3x \sin 2x - 9\cos x \cos 2x \sin 3x - 5\sin x \cos 2x \cos 3x$; b. $W_S(\pi/4) = -8$, so S is linearly independent.
2. a. $W_S(x) = -e^{2x}$ b. $W_S(0) = -1$, so S is linearly independent.
3. a. $W_S(x) = -ne^{2kx}$ b. $W_S(x) = -n \neq 0$, so S is linearly independent.
4. a. $W_S(x) = z(x)$ b. S is linearly dependent.
5. a. $W_S(x) = z(x)$ b. S is linearly dependent.
6. a. $W_S(x) = 18\cos^2 x \cos^2 2x + 18\cos^2 x \sin^2 2x + 18\sin^2 x \cos^2 2x + 18\sin^2 x \sin^2 2x$
   b. $W_S(0) = 18$, so S is linearly independent.
7. a. $W_S(x) = 12(\tan x)\sec^2(2x)\sec^2(3x)[3(\tan 3x) - 2(\tan 2x)]$
   $+ 6\tan 2x \sec^2(3x)\sec^2(x)[(\tan x) - 3(\tan 3x)]$
   $+ 4\tan 3x \sec^2(2x)\sec^2(x)[2(\tan 2x) - (\tan x)]$
   b. $W_S(\pi/3) = 216$, so S is linearly independent.
8. a. $W_S(x) = \frac{23}{160}x^{-\frac{3}{20}}$ b. $W_S(1) = \frac{23}{160}$, so S is linearly independent.
9. a. $W_S(x) = \frac{1}{144\,000}x^{-\frac{283}{60}}$ b. $W_S(1) = \frac{1}{144\,000}$, so S is linearly independent.
10. a. $W_S(x) = \left(-\frac{9}{16}\right)\frac{1}{[(x-1)(x-2)(x-3)(x-4)]^{3/2}}$ b. $W_S(5) \neq 0$, so S is linearly independent.
11. a. $W_S(x) = (5^x)(4^x)(3^x)(\ln 4 - \ln 3)(\ln 5 - \ln 3)(\ln 5 - \ln 4)$ b. $W_S(0) = (\ln 4 - \ln 3)(\ln 5 - \ln 3)(\ln 5 - \ln 4) \neq 0$, so S is linearly independent.
12. a. $W_S(x) = z(x)$ b. S is linearly dependent.
13. a. $\{e^{k_1 x}, e^{k_2 x}, \ldots, e^{k_n x}\}$ b. $W_{S'}(x) = V(k_1, k_2, \ldots, k_n) \cdot e^{(k_1+k_2+\cdots+k_n)x}$ c. $W_{S'}(0) = V(k_1, k_2, \ldots, k_n) \neq 0$, since the k_i are distinct, so S is linearly independent.
14. a. $\{b_1^x, b_2^x, \ldots, b_n^x\}$ b. $W_{S'}(x) = V(\ln(b_1), \ln(b_2), \ldots, \ln(b_n))b_1^x \cdot b_2^x \cdot \cdots \cdot b_n^x$ c. $W_{S'}(0) = V(\ln(b_1), \ln(b_2), \ldots, \ln(b_n)) \neq 0$, since the b_i are distinct, so S is linearly independent.
15. a. $\{x^{k_1}, x^{k_2}, \ldots, x^{k_n}\}$ b. $W_{S'}(x) = V(k_1, k_2, \ldots, k_n)x^{k_1+k_2+\cdots+k_n-n(n+1)/2}$ c. $W_{S'}(0) = V(k_1, k_2, \ldots, k_n) \neq 0$, since the k_i are distinct, so S is linearly independent.
16. a. $\{(x-k_1)^m, (x-k_2)^m, \ldots, (x-k_n)^m\}$ b. and c. If m is a positive integer and $n > m+1$, then $W_{S'}(x) = z(x)$ and consequently S will be dependent, since $\dim(\mathbb{R}^m) = m+1$, and S' contains $n > m+1$ vectors from $\mathbb{R}^m$. If m is not a positive integer, then m, $m-1$, $\ldots$, $m-i$ is never zero for any positive integer i, and we get:
   $W_S(x) = \pm m \cdot m(m-1) \cdot m(m-1)(m-2) \cdot \cdots \cdot (m-n+2) \cdot$
   $(x-k_1)^{m+n-1}(x-k_2)^{m+n-1} \cdot \cdots \cdot (x-k_n)^{m+n-1} \cdot V(x-k_1, x-k_2, \ldots, x-k_n)$;
   Note: the sign + or − depends on the remainder j when n is divide by 4, i.e. $n = 4i + j$, where i is a non-negative integer and $j = 0, 1, 2,$ or 3, since we will need to perform row

exchanges in order to bring the Wronskian matrix into a form similar to the Vandermonde matrix (note that the powers of $x - k_i$ are in decreasing rather than increasing order); the number of these exchanges depends on $j;$ by letting x be any number bigger than k_n (where we assume the k_i are in increasing order), we get a non-zero value for $W_{S'}(x),$ so S is independent.

Chapter Six Exercises

6.1 Exercises

1. $p(\lambda) = \lambda^2 + \lambda - 6$; $Eig(A,2) = Span(\{\langle -1,1\rangle\})$; $Eig(A,-3) = Span(\{\langle -2,1\rangle\})$. Each is 1-dimensional.
2. $p(\lambda) = \lambda^2 - 8\lambda + 15$; $Eig(A,5) = Span(\{\langle 2,5\rangle\})$; $Eig(A,3) = Span(\{\langle 1,2\rangle\})$. Each is 1-dimensional.
3. $p(\lambda) = \lambda^2 - 11\lambda - 12$; $Eig(A,-1) = Span(\{\langle -2,3\rangle\})$; $Eig(A,12) = Span(\{\langle 3,2\rangle\})$. Each is 1-dimensional.
4. $p(\lambda) = \lambda^2 + 3\lambda - 10$; $Eig(A,2) = Span(\{\langle -4,3\rangle\})$; $Eig(A,-5) = Span(\{\langle -3,2\rangle\})$. Each is 1-dimensional.
5. $p(\lambda) = \lambda^2 + 36$; since the eigenvalues are imaginary, there are no eigenvectors.
6. $p(\lambda) = \lambda^2 - 15\lambda + 44$; $Eig(A,4) = Span(\{\langle 5,2\rangle\})$; $Eig(A,11) = Span(\{\langle 7,3\rangle\})$. Each is 1-dimensional.
7. $p(\lambda) = (\lambda-5)(\lambda+2)(\lambda+4)$; $Eig(A,5) = Span(\{\langle 1,0,0\rangle\})$;
   $Eig(A,-2) = Span(\{\langle 4,-7,0\rangle\})$;
   $Eig(A,-4) = Span(\{\langle 2,27,18\rangle\})$. Each is 1-dimensional.
8. $p(\lambda) = (\lambda-4)(\lambda-7)(\lambda+2)$; $Eig(A,4) = Span(\{\langle 6,2,1\rangle\})$;
   $Eig(A,7) = Span(\{\langle 0,-3,2\rangle\})$;
   $Eig(A,-2) = Span(\{\langle 0,0,1\rangle\})$. Each is 1-dimensional.
9. $p(\lambda) = \lambda(\lambda+5)(\lambda-8)$; $Eig(A,0) = Span(\{\langle 3,-5,0\rangle\})$; $Eig(A,-5) = Span(\{\langle 1,0,0\rangle\})$;
   $Eig(A,8) = Span(\{\langle 69,-91,104\rangle\})$. Each is 1-dimensional.
10. $p(\lambda) = (\lambda-3)^2(\lambda-2)(\lambda-4)$; $Eig(A,3) = Span(\{\langle 1,0,0,0\rangle,\langle 0,5,1,0\rangle\})$, 2-dimensional;
    $Eig(A,2) = Span(\{\langle -3,1,0,0\rangle\})$; $Eig(A,4) = Span(\{\langle 27,-9,-2,2\rangle\})$; the other two are 1-dimensional.
11. $p(\lambda) = (\lambda+2)^2(\lambda-3)^2$; $Eig(A,-2) = Span(\{\langle 5,4,0,0\rangle,\langle 0,0,-1,3\rangle\})$;
    $Eig(A,3) = Span(\{\langle 0,-1,2,0\rangle,\langle 0,0,0,1\rangle\})$. Each is 2-dimensional.
12. $p(\lambda) = (\lambda-5)^3(\lambda+3)$; $Eig(A,5) = Span(\{\langle 0,0,1,0\rangle,\langle 0,0,0,1\rangle,\langle 8,7,0,0\rangle\})$, 3-dimensional, and
    $Eig(A,-3) = Span(\{\langle 0,1,-3,2\rangle\})$, 1-dimensional.
13. $p(\lambda) = \lambda^2 - \lambda - 10/9$; $Eig(A,5/3) = Span(\{\langle -7,4\rangle\})$; $Eig(A,-2/3) = Span(\{\langle 1,-1\rangle\})$. Each is 1-dimensional.
14. $p(\lambda) = (\lambda+1/3)(\lambda-4/3)(\lambda-2/3)$; $Eig(A,-1/3) = Span(\{\langle 1,0,0\rangle\})$;
    $Eig(A,4/3) = Span(\{\langle 1,1,0\rangle\})$;
    $Eig(A,2/3) = Span(\{\langle 3,1,2\rangle\})$. Each is 1-dimensional.
15. $p(\lambda) = \lambda(\lambda-5/2)(\lambda+3/2)(\lambda-1/2)$; $Eig(A,5/2) = Span(\{\langle 1,0,0,0\rangle\})$;
    $Eig(A,0) = Span(\{\langle 7,5,0,0\rangle\})$;
    $Eig(A,-3/2) = Span(\{\langle 11,12,-4,0\rangle\})$; $Eig(A,1/2) = Span(\{\langle 57,34,10,-8\rangle\})$. Each is 1-dimensional.
16. a. $p(\lambda) = \lambda^2 - 36$; $Eig(A,6) = Span(\{\langle 3,2\rangle\})$; $Eig(A,-6) = Span(\{\langle -3,2\rangle\})$.
    b. the eigenvalues are imaginary: $\pm 6i$, so there are no eigenvectors.
17. a. $p(\lambda) = (\lambda-3)^2(\lambda+2)$; $Eig(A,3) = Span(\{\langle 1,0,0\rangle,\langle 0,2,5\rangle\})$, 2-dimensional;
    $Eig(A,-2) = Span(\{\langle -3,1,0\rangle\})$, 1-dimensional.
    b. $p(\lambda) = (\lambda-3)^2(\lambda+2)$; $Eig(A,3) = Span(\{\langle 1,0,0\rangle\})$;

$Eig(A,-2) = Span(\{\langle -14,5,0\rangle\})$; both 1-dimensional.

18. a. $p(\lambda) = (\lambda+7)^2(\lambda-2)$; $Eig(A,-7) = Span(\{\langle 3,1,0\rangle,\langle 0,0,1\rangle\})$, 2-dimensional; $Eig(A,2) = Span(\{\langle 0,1,-2\rangle\})$, 1-dimensional.
 b. $p(\lambda) = (\lambda+7)^2(\lambda-2)$; $Eig(A,-7) = Span(\{\langle 0,0,1\rangle\})$; $Eig(A,2) = Span(\{\langle 0,1,2\rangle\})$; each is 1-dimensional.

19. a. $p(\lambda) = (\lambda-3)^2(\lambda+2)^2$; $Eig(A,-2) = Span(\{\langle 1,0,0,0\rangle\})$, 1-dimensional; $Eig(A,3) = Span(\{\langle -2,1,0,0\rangle,\langle 49,0,15,5\rangle\})$, 2-dimensional.
 b. $p(\lambda) = (\lambda-3)^2(\lambda+2)^2$; $Eig(A,-2) = Span(\{\langle 1,0,0,0\rangle,\langle 0,7,5,0\rangle\})$; $Eig(A,3) = Span(\{\langle -2,1,0,0\rangle,\langle 46,0,15,5\rangle\})$. Both are 2-dimensional.
 c. $p(\lambda) = (\lambda-3)^2(\lambda+2)^2$; $Eig(A,-2) = Span(\{\langle 1,0,0,0\rangle,\langle 0,7,5,0\rangle\})$, 2-dimensional; $Eig(A,3) = Span(\{\langle -2,1,0,0\rangle\})$, 1-dimensional.

20. a. $p(\lambda) = (\lambda-3)(\lambda+2)^3$; $Eig(A,-2) = Span(\{\langle 1,0,0,0\rangle\})$; $Eig(A,3) = Span(\{\langle -2,1,0,0\rangle\})$. Both are 1-dimensional.
 b. $p(\lambda) = (\lambda-3)(\lambda+2)^3$; $Eig(A,3) = Span(\{\langle -4,1,0,0\rangle\})$, 1-dimensional; $Eig(A,-2) = Span(\{\langle 1,0,0,0\rangle,\langle 0,2,5,0\rangle\})$, 2-dimensional.
 c. $p(\lambda) = (\lambda-3)(\lambda+2)^3$; $Eig(A,3) = Span(\{\langle -4,1,0,0\rangle\})$, 1-dimensional; $Eig(A,-2) = Span(\{\langle 1,0,0,0\rangle,\langle 0,2,5,0\rangle,\langle 0,-4,0,5\rangle\})$, 3-dimensional.

21. a. $p(\lambda) = (\lambda-3)^2(\lambda-1)^3$; $Eig(A,1) = Span(\{\langle 1,0,0,0,0\rangle,\langle 0,3,1,0,0\rangle\})$; $Eig(A,3) = Span(\{\langle 2,1,0,0,0\rangle,\langle 0,0,3,1,0\rangle\})$. Both are 2-dimensional.
 b. $p(\lambda) = (\lambda-3)^2(\lambda-1)^3$;
 $Eig(A,1) = Span(\{\langle 1,0,0,0,0\rangle,\langle 0,3,1,0,0\rangle,\langle 0,0,0,-5,2\rangle\})$, 3-dimensional; $Eig(A,3) = Span(\{\langle 2,1,0,0,0\rangle,\langle 0,0,3,1,0\rangle\})$, 2-dimensional.
 c. $p(\lambda) = (\lambda-3)^2(\lambda-1)^3$;
 $Eig(A,1) = Span(\{\langle 1,0,0,0,0\rangle,\langle 0,3,1,0,0\rangle,\langle 0,5,0,-5,2\rangle\})$, 3-dimensional; $Eig(A,3) = Span(\{\langle 2,1,0,0,0\rangle\})$, 1-dimensional.

22. a. $p(\lambda) = \left(\lambda-\sqrt{3}\right)^2\left(\lambda-\sqrt{2}\right)$; $Eig\left(A,\sqrt{3}\right) = Span(\{\langle 1,1,0\rangle,\langle 0,0,1\rangle\})$, 2-dimensional;
 $Eig\left(A,\sqrt{2}\right) = Span\left(\left\{\left\langle 0,\sqrt{3}-\sqrt{2},5\right\rangle\right\}\right)$, 1-dimensional.
 b. $p(\lambda) = \left(\lambda-\sqrt{3}\right)^2\left(\lambda-\sqrt{2}\right)$; $Eig\left(A,\sqrt{3}\right) = Span(\{\langle 0,0,1\rangle\})$, 1-dimensional; $Eig\left(A,\sqrt{2}\right) = Span\left(\left\{\left\langle 0,\sqrt{3}-\sqrt{2},5\right\rangle\right\}\right)$, 1-dimensional.

23. a. $p(\lambda) = (\lambda-3\pi^2)^2(\lambda-2\pi)$; $Eig(A,3\pi^2) = Span(\{\langle 0,0,1\rangle\})$, 1-dimensional; $Eig(A,2\pi) = Span(\{\langle 0,3\pi-2,1\rangle\})$, 1-dimensional.
 b. $p(\lambda) = (\lambda-3\pi^2)^2(\lambda-2\pi)$; $Eig(A,3\pi^2) = Span(\{\langle \pi,2,0\rangle,\langle 0,0,1\rangle\})$, 2-dimensional; $Eig(A,2\pi) = Span(\{\langle 0,3\pi-2,1\rangle\})$, 1-dimensional.

24. $A^{\top} = \begin{bmatrix} -8 & 5 \\ -10 & 7 \end{bmatrix}$. We get the same characteristic polynomials and thus same eigenvalues.
 However, for $A^{\top}$, $Eig(A^{\top},-3) = Span(\{\langle 1,1\rangle\})$ and $Eig(A^{\top},2) = Span(\{\langle 1,2\rangle\})$. These eigenspaces are different from the eigenspaces for A. Notice, however, that the corresponding eigenspaces are orthogonal to each other!

31. a. $\lambda^2 - 2\cos(\theta)\lambda + 1$; b. The discriminant is $-4\sin^2(\theta)$, which is negative unless $\sin(\theta) = 0$, which corresponds to $\theta = \pi n$. In this case, $\lambda = \cos(n\pi) = \pm 1$, and $R_\theta = \pm I$.

32. a. $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ b. rotate a vector $\vec{v}$ counterclockwise by θ then reflect this resulting

vector across the y-axis;
c. $\lambda^2 - 1$ d. the eigenvalues are always $\lambda = 1$ and $\lambda = -1$;
e. $Eig(A,-1) = Span(\{\langle \sin(\theta), 1+\cos(\theta)\rangle\})$ and
$Eig(A,1) = Span(\{\langle \sin(\theta), -1+\cos(\theta)\rangle\})$.
f. $Eig(A,-1) = Span(\{\langle \sin(\theta/2), \cos(\theta/2)\rangle\})$ and
$Eig(A,1) = Span(\{\langle \cos(\theta/2), -\sin(\theta/2)\rangle\})$.
h. they are orthogonal to each other!
i. $\begin{bmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{bmatrix}$; $Eig(A,-1) = Span(\{\langle -2,3\rangle\})$ and $Eig(A,1) = Span(\{\langle 3,2\rangle\})$.
j. Repeat (a) to (h) for the matrix B:
a. $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ b. reflect $\vec{v}$ across the x-axis, then rotate this resulting vector counterclockwise by θ.
c. $\lambda^2 - 1$ d. the eigenvalues are always $\lambda = 1$ and $\lambda = -1$;
e. $Eig(A,-1) = Span(\{\langle \sin(\theta), -1-\cos(\theta)\rangle\})$ and
$Eig(A,1) = Span(\{\langle \sin(\theta), 1-\cos(\theta)\rangle\})$.
f. $Eig(A,-1) = Span(\{\langle \sin(\theta/2), -\cos(\theta/2)\rangle\})$ and
$Eig(A,1) = Span(\{\langle \cos(\theta/2), \sin(\theta/2)\rangle\})$.
h. again, they are orthogonal to each other.
i. $\begin{bmatrix} -5/13 & 12/13 \\ 12/13 & 5/13 \end{bmatrix}$; $Eig(A,-1) = Span(\{\langle -3,2\rangle\})$ and $Eig(A,1) = Span(\{\langle 2,3\rangle\})$.

34. b. $Eig(A_1 \oplus A_2,-5) = Span(\{\langle 0, 0, 1, 2, 1\rangle\})$;
$Eig(A_1 \oplus A_2,3) = Span(\{\langle -2, 1, 0, 0, 0\rangle, \langle 0, 0, -1, 1, 0\rangle, \langle 0, 0, 1, 0, 1\rangle\})$;
$Eig(A_1 \oplus A_2,7) = Span(\{\langle -5, 2, 0, 0, 0\rangle\})$

6.2 Exercises

1. For $\lambda = -5 : \{\langle 1,1,0\rangle\}$; for $\lambda = 3 : \{\langle 1,1,1\rangle\}$ and for $\lambda = 7 : \{\langle 0,-1,1\rangle\}$.
2. Hint: the exponent of p_1 can be 0, 1, 2, …, n_1.
3. 24 possibilities: $\pm\{1,2,3,5,6,9,10,15,18,30,45,90\}$; roots are: $\lambda = 5,6,-3$
4. 24 possibilities: $\pm\{1,2,3,4,6,8,9,12,18,24,36,72\}$; roots are: $\lambda = 6,-3,-4$
5. 8 possibilities: $\pm\{1,3,5,15\}$; roots are: $\lambda = 5, 3+\sqrt{6}, 3-\sqrt{6}$.
6. $p(\lambda) = \lambda^3 - 9\lambda^2 + 23\lambda - 15 = (\lambda-5)(\lambda-1)(\lambda-3)$; for $\lambda = 1 : \{\langle -1,0,1\rangle\}$, $dim = 1$; for $\lambda = 3 : \{\langle 1,0,1\rangle\}$, $dim = 1$; for $\lambda = 5 : \{\langle 0,1,0\rangle\}$, $dim = 1$.
7. $p(\lambda) = \lambda^3 - 2\lambda^2 - 15\lambda + 36 = (\lambda-3)^2(\lambda+4)$;
for $\lambda = 3 : \{\langle 0,0,1\rangle\}$, $dim = 1$; for $\lambda = -4 : \{\langle -7,0,1\rangle\}$, $dim = 1$.
8. $p(\lambda) = \lambda^3 - 15\lambda^2 + 72\lambda - 112 = (\lambda-7)(\lambda-4)^2$;
for $\lambda = 7 : \{\langle 1,1,1\rangle\}$, $dim = 1$; for $\lambda = 4 : \{\langle -1,1,0\rangle, \langle -1,0,1\rangle\}$, $dim = 2$.
9. $p(\lambda) = \lambda^3 - 5\lambda^2 - 7\lambda + 35$; for $\lambda = 5 : \{\langle 0,0,1\rangle\}$, $dim = 1$;
for $\lambda = \sqrt{7} : \{\langle 1, \sqrt{7}-3, 0\rangle\}$, $dim = 1$; for $\lambda = -\sqrt{7} : \{\langle 1, -\sqrt{7}-3, 0\rangle\}$, $dim = 1$.
10. $p(\lambda) = \lambda^3 - 3\lambda^2 - 10\lambda + 24 = (\lambda-2)(\lambda-4)(\lambda+3)$; for $\lambda = -3 : \{\langle 2,9,2\rangle\}$, $dim = 1$; for $\lambda = 4 : \{\langle 1,1,1\rangle\}$, $dim = 1$; for $\lambda = 2 : \{\langle 36,42,31\rangle\}$, $dim = 1$.
11. $p(\lambda) = \lambda^3 - 7/4\lambda^2 + 7/16\lambda + 15/64$; for $\lambda = -1/4 : \{\langle 2,3,2\rangle\}$, $dim = 1$; for $\lambda = 3/4 : \{\langle 1,1,1\rangle\}$, $dim = 1$; for $\lambda = 5/4 : \{\langle 4,4,3\rangle\}$, $dim = 1$.

12. $p(\lambda) = \lambda^3 - 13\lambda - 12$; for $\lambda = 4 : \{\langle 0,-1,1\rangle\}$, $dim = 1$; for $\lambda = -1 : \{\langle -1,0,1\rangle\}$, $dim = 1$; for $\lambda = -3 : \{\langle 2,-3,0\rangle\}$, $dim = 1$.
13. $p(\lambda) = \lambda^3 - 15\lambda^2 + 72\lambda - 112$; for $\lambda = 4 : \{\langle 4,0,5\rangle, \langle 2,-5,0\rangle\}$, $dim = 2$; for $\lambda = 7 : \{\langle 1,-1,1\rangle\}$, $dim = 1$.
14. $p(\lambda) = \lambda^3 - 15\lambda^2 + 72\lambda - 112$ (note: same as Exercise 13); for $\lambda = 4 : \{\langle 2,-1,1\rangle\}$, $dim = 1$; for $\lambda = 7 : \{\langle 1,-1,1\rangle\}$, $dim = 1$.
15. $p(\lambda) = \lambda^3 + \lambda^2 - 21\lambda - 45$; for $\lambda = 5 : \{\langle -4,2,1\rangle\}$, $dim = 1$; for $\lambda = -3 : \{\langle -2,1,0\rangle, \langle 1,0,1\rangle\}$, $dim = 2$;
16. $p(\lambda) = \lambda^3 - 5\lambda^2 - 32\lambda - 36$; for $\lambda = 9 : \{\langle 1,4,-2\rangle\}$, $dim = 1$; for $\lambda = -2 : \{\langle 1,1,0\rangle, \langle 1,0,1\rangle\}$, $dim = 2$;
17. $p(\lambda) = \lambda^3 - 7\lambda^2 - 5\lambda + 75$; for $\lambda = -3 : \{\langle 1,3,-2\rangle\}$, $dim = 1$; for $\lambda = 5 : \{\langle 0,1,-1\rangle, \langle 2,3,0\rangle\}$, $dim = 2$;
18. $p(\lambda) = \lambda^3 + \frac{1}{3}\lambda^2 - \frac{40}{9}\lambda - \frac{112}{27}$; for $\lambda = 7/3 : \{\langle 1,1,2\rangle\}$, $dim = 1$; for $\lambda = -4/3 : \{\langle -2,5,0\rangle, \langle 3,0,5\rangle\}$, $dim = 2$;
19. $p(\lambda) = \lambda^3 + \frac{1}{4}\lambda^2 - \frac{33}{16}\lambda + \frac{63}{64}$; for $\lambda = -7/4 : \{\langle -2,-1,2\rangle\}$, $dim = 1$; for $\lambda = 3/4 : \{\langle 3,1,0\rangle, \langle 3,0,5\rangle\}$, $dim = 2$;
20. $p(\lambda) = \lambda^3 - \frac{2}{5}\lambda^2 - \frac{3}{5}\lambda + \frac{36}{125}$; for $\lambda = -4/5 : \{\langle -1,-1,2\rangle\}$, $dim = 1$; for $\lambda = 3/5 : \{\langle 2,1,0\rangle, \langle 3,0,5\rangle\}$, $dim = 2$;
21. $p(\lambda) = \lambda^4 - 25\lambda^2 - 3\lambda^3 + 75\lambda$; for $\lambda = -5 : \{\langle 3,0,-5,4\rangle\}$, $dim = 1$; for $\lambda = 0 : \{\langle -4,0,0,3\rangle\}$, $dim = 1$; for $\lambda = 3 : \{\langle 0,1,0,0\rangle\}$, $dim = 1$; for $\lambda = 5 : \{\langle 3,0,5,4\rangle\}$, $dim = 1$.
22. $p(\lambda) = \lambda^4 - 98\lambda^2 + 2401 = (\lambda - 7)^2(\lambda + 7)^2$; for $\lambda = -7 : \{\langle -1,0,0,1\rangle, \langle 0,-1,1,0\rangle\}$, $dim = 2$; for $\lambda = 7 : \{\langle 1,0,0,1\rangle, \langle 0,1,1,0\rangle\}$, $dim = 2$.
23. $p(\lambda) = \lambda^4 - 116\lambda^2 + 1600 = (\lambda + 10)(\lambda - 4)(\lambda + 4)(\lambda - 10)$; for $\lambda = -10 : \{\langle 1,-1,1,-1\rangle\}$, $dim = 1$; for $\lambda = 10 : \{\langle 1,1,1,1\rangle\}$, $dim = 1$; for $\lambda = -4 : \{\langle -1,-1,1,1\rangle\}$, $dim = 1$; for $\lambda = 4 : \{\langle -1,1,1,-1\rangle\}$, $dim = 1$.
24. $p(\lambda) = \lambda^4 - 7\lambda^3 + \lambda^2 + 63\lambda - 90 = (\lambda - 2)(\lambda - 3)(\lambda + 3)(\lambda - 5)$; for $\lambda = -3 : \{\langle 0,-1,0,1\rangle\}$, $dim = 1$; for $\lambda = 3 : \langle 3,0,1,0\rangle$, $dim = 1$; for $\lambda = 2 : \{\langle -9,1,-3,-2\rangle\}$, $dim = 1$; for $\lambda = 5 : \{\langle -1,-1,0,1\rangle\}$, $dim = 1$.
25. $p(\lambda) = \lambda^4 - 3\lambda^3 - 12\lambda^2 + 20\lambda + 48 = (\lambda - 3)(\lambda - 4)(\lambda + 2)^2$; for $\lambda = 3 : \{\langle 2,-1,1,-1\rangle\}$, $dim = 1$; for $\lambda = 4 : \{\langle -5,2,-2,0\rangle\}$, $dim = 1$; for $\lambda = -2 : \{\langle 3,0,3,-1\rangle, \langle -3,3,0,1\rangle\}, dim = 2$.
26. $p(\lambda) = \lambda^4 + 2\lambda^3 - 23\lambda^2 - 24\lambda + 144 = (\lambda - 3)^2(\lambda + 4)^2$; for $\lambda = 3 : \{\langle 1,0,2,0\rangle, \langle 0,1,-2,1\rangle\}$, $dim = 2$; for $\lambda = -4 : \{\langle -2,1,-5,1\rangle\}$, $dim = 1$.
27. $p(\lambda) = \lambda^4 - \lambda^3 - 18\lambda^2 + 52\lambda - 40 = (\lambda - 2)^3(\lambda + 5)$; for $\lambda = 2 : \{\langle 1,0,2,0\rangle, \langle 0,1,3,0\rangle, \langle 0,0,-3,1\rangle\}$, $dim = 3$; for $\lambda = -5 : \{\langle -2,1,-3,1\rangle\}$, $dim = 1$.
28. $p(\lambda) = \lambda^4 - 5\lambda^3 + 6\lambda^2 + 4\lambda - 8 = (\lambda - 2)^3(\lambda + 1)$; for $\lambda = 2 : \{\langle -2,5,1,0\rangle, \langle -5,10,0,4\rangle\}$, $dim = 2$; for $\lambda = -1 : \{\langle -2,1,-3,7\rangle\}$, $dim = 1$.
29. $p(\lambda) = \lambda^5 - 10\lambda^4 + 32\lambda^3 - 32\lambda^2$; for $\lambda = 2 : \{\langle 0,0,1,0,0\rangle\}$, $dim = 1$. for $\lambda = 0 : \{\langle 0,-1,0,1,0\rangle, \langle 1,0,0,0,-1\rangle\}$, $dim = 2$; for $\lambda = 4 : \{\langle 0,1,0,1,0\rangle, \langle 1,0,0,0,1\rangle\}$, $dim = 2$.
30. $p(\lambda) = \lambda^3 + \lambda^2 - 31\lambda + 46$; $Eig(A, -6.6758) = Span(\{\langle -0.60015, -0.8689, 1\rangle\})$; $Eig(A, 1.7594) = Span(\{\langle 1.10664, 0.3865, 1\rangle\})$; $Eig(A, 3.9164) = Span(\{\langle -1.72078, 2.3395, 1\rangle\})$.
31. $p(\lambda) = \lambda^3 + 8\lambda^2 + 7\lambda - 13$; $Eig(A, -6.6545) = Span(\{\langle -0.5515, 0.1185, 1\rangle\})$;

$Eig(A, -2.2239) = Span(\{\langle 0.92536, -4.1326, 1\rangle\})$;
$Eig(A, 0.87843) = Span(\{\langle 1.9595, 0.680745, 1\rangle\})$.

32. $p(\lambda) = \lambda^4 - 3\lambda^3 - 14\lambda^2 + 26\lambda + 10$;
$Eig(A, -3.3149) = Span(\{\langle -0.158774, 0.156133, -0.072369, 1\rangle\})$;
$Eig(A, -0.33044) = Span(\{\langle -2.75815, -5.4277, 8.15926, 1\rangle\})$;
$Eig(A, 1.9403) = Span(\{\langle 6.3174, 1.3771, 2.929, 1\rangle\})$;
$Eig(A, 4.705) = Span(\{\langle 2.3495, -5.35553, -2.89094, 1\rangle\})$.

37. a. False. b. True. c. False. d. False. e. True. f. False. g. True.

6.3 Exercises

Note: the diagonal entries of D can be rearranged, as long as the corresponding eigenvectors are also located in the corresponding columns of C.

1. $D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$; $C = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$ 2. $D = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$; $C = \begin{bmatrix} -3 & -4 \\ 2 & 3 \end{bmatrix}$

3. This matrix is not diagonalizable because the eigenvalues are imaginary.

4. $D = \begin{bmatrix} -2/3 & 0 \\ 0 & 5/3 \end{bmatrix}$; $C = \begin{bmatrix} 1 & -7 \\ -1 & 4 \end{bmatrix}$ 5. $D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$;
$C = \begin{bmatrix} 2 & 4 & 1 \\ 27 & -7 & 0 \\ 18 & 0 & 0 \end{bmatrix}$

6. $D = \begin{bmatrix} -1/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$ 7. $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$;
$C = \begin{bmatrix} -3 & 1 & 0 & 27 \\ 1 & 0 & 5 & -9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

8. $D = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ 9. $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$;
$C = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

10. This matrix is not diagonalizable because there are only ***two*** linearly independent vectors, and this is a 3×3 matrix.

11. $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$; $C = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ 12. $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$;

$C = \begin{bmatrix} 2 & 36 & 1 \\ 9 & 42 & 1 \\ 2 & 31 & 1 \end{bmatrix}$

13. $D = \begin{bmatrix} -1/4 & 0 & 0 \\ 0 & 3/4 & 0 \\ 0 & 0 & 5/4 \end{bmatrix}$; $C = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 4 \\ 2 & 1 & 3 \end{bmatrix}$ 14. $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$;

$C = \begin{bmatrix} 2 & -1 & 0 \\ -3 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

15. $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$; $C = \begin{bmatrix} 4 & 2 & 1 \\ 0 & -5 & -1 \\ 5 & 0 & 1 \end{bmatrix}$

16. This matrix is not diagonalizable because there are only ***two*** linearly independent vectors, and this is a 3×3 matrix.

17. $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$; $C = \begin{bmatrix} -2 & 1 & -4 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ 18. $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$;

$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & -1 & -2 \end{bmatrix}$

19. $D = \begin{bmatrix} -7 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$; $C = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ 20. $D = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$;

$C = \begin{bmatrix} 0 & -9 & 3 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & -3 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}$

21. $D = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$; $C = \begin{bmatrix} 3 & -3 & 2 & -5 \\ 0 & 3 & -1 & 2 \\ 3 & 0 & 1 & -2 \\ -1 & 1 & -1 & 0 \end{bmatrix}$

22. This matrix is not diagonalizable because there are only ***three*** linearly independent vectors, and this is a 4×4 matrix.

23. $D = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$; $C = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 2 & 3 & -3 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

24. $D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$

25. Only the matrix in (a) is diagonalizable.
26. Only the matrix in (a) is diagonalizable.
27. Only the matrix in (a) is diagonalizable.
28. Only the matrix in (b) is diagonalizable.
29. Only the matrix in (c) is diagonalizable.
30. Only the matrix in (b) is diagonalizable.
31. Only the matrix in (a) is diagonalizable.
32. Only the matrix in (b) is diagonalizable.

33. $\begin{bmatrix} -28381 & -37884 \\ 18942 & 25288 \end{bmatrix}$

34. $\begin{bmatrix} \frac{22003}{729} & \frac{22099}{729} \\ -\frac{12628}{729} & -\frac{12724}{729} \end{bmatrix}$

35. $\begin{bmatrix} 3125 & 1804 & -3167 \\ 0 & -32 & -1488 \\ 0 & 0 & -1024 \end{bmatrix}$

36. $\begin{bmatrix} 243 & -1477 & 1261 & 14472 \\ 0 & 32 & 1804 & -2996 \\ 0 & 0 & -3125 & -461 \\ 0 & 0 & 0 & 1024 \end{bmatrix}$

37. $\begin{bmatrix} -6493 & 3368 & 3368 \\ -23300 & 20175 & 3368 \\ 16564 & -16564 & 243 \end{bmatrix}$

38. $\begin{bmatrix} -5322 & -362 & 6708 \\ -4749 & -605 & 6378 \\ -5354 & -362 & 6740 \end{bmatrix}$

39. $\begin{bmatrix} 483 & 484 & 484 \\ -3801 & -2777 & -3801 \\ 3075 & 2050 & 3074 \end{bmatrix}$ 40. $\begin{bmatrix} -77891 & -31566 & 63132 \\ 78915 & 32590 & -63132 \\ -78915 & -31566 & 64156 \end{bmatrix}$

41. $\begin{bmatrix} -59113 & 59081 & 59081 \\ -236324 & 236292 & 236324 \\ 118162 & -118162 & -118194 \end{bmatrix}$

42. $\begin{bmatrix} 0 & 0 & 0 & 16807 \\ 0 & 0 & 16807 & 0 \\ 0 & 16807 & 0 & 0 \\ 16807 & 0 & 0 & 0 \end{bmatrix}$ 43. $\begin{bmatrix} 3125 & -1899 & -8646 & -1899 \\ 3368 & -518 & -10104 & -275 \\ 0 & -633 & 243 & -633 \\ -3368 & 550 & 10104 & 307 \end{bmatrix}$

44. $\begin{bmatrix} 7448 & 8030 & -8030 & -1650 \\ -3212 & -3519 & 3487 & 825 \\ 3212 & 3487 & -3519 & -825 \\ -1100 & -1375 & 1375 & 793 \end{bmatrix}$ 45. $\begin{bmatrix} -12596 & -18942 & 6314 & 18942 \\ 6314 & 9503 & -3157 & -9471 \\ -18942 & -28413 & 9503 & 28413 \\ 6314 & 9471 & -3157 & -9439 \end{bmatrix}$

46. $\begin{bmatrix} 512 & 0 & 0 & 0 & 512 \\ 0 & 512 & 0 & 512 & 0 \\ 0 & 0 & 32 & 0 & 0 \\ 0 & 512 & 0 & 512 & 0 \\ 512 & 0 & 0 & 0 & 512 \end{bmatrix}$ 47. $\begin{bmatrix} 0 & 11664 \\ 5184 & 0 \end{bmatrix}$ 48. $\begin{bmatrix} 243 & 825 & -330 \\ 0 & -32 & 110 \\ 0 & 0 & 243 \end{bmatrix}$

49. $\begin{bmatrix} -16807 & 0 & 0 \\ -5613 & 32 & 0 \\ 11226 & -33678 & -16807 \end{bmatrix}$ 50. $\begin{bmatrix} -32 & -550 & 770 & 220 \\ 0 & 243 & -385 & 1155 \\ 0 & 0 & -32 & 825 \\ 0 & 0 & 0 & 243 \end{bmatrix}$

51. $\begin{bmatrix} -32 & -1100 & 440 & -880 \\ 0 & 243 & -110 & 220 \\ 0 & 0 & -32 & 0 \\ 0 & 0 & 0 & -32 \end{bmatrix}$ 52. $\begin{bmatrix} 1 & 484 & -1452 & 4356 & 10890 \\ 0 & 243 & -726 & 2178 & 5445 \\ 0 & 0 & 1 & 726 & 1815 \\ 0 & 0 & 0 & 243 & 605 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

61. a. False b. False c. False d. True e. False f. True g. False h. True i. True j. False.

6.4 Exercises

We provide the answers for e^{tA}. To get e^{A}, just replace t with 1.

1. $$\begin{bmatrix} -8e^{2t}+9e^{-5t} & -12e^{2t}+12e^{-5t} \\ 6e^{2t}-6e^{-5t} & 9e^{2t}-8e^{-5t} \end{bmatrix}$$

2. $$\begin{bmatrix} -\frac{4}{3}e^{-\frac{2}{3}t}+\frac{7}{3}e^{\frac{5}{3}t} & -\frac{7}{3}e^{-\frac{2}{3}t}+\frac{7}{3}e^{\frac{5}{3}t} \\ \frac{4}{3}e^{-\frac{2}{3}t}-\frac{4}{3}e^{\frac{5}{3}t} & \frac{7}{3}e^{-\frac{2}{3}t}-\frac{4}{3}e^{\frac{5}{3}t} \end{bmatrix}$$

3. $$\begin{bmatrix} e^{5t} & -\frac{4}{7}e^{-2t}+\frac{4}{7}e^{5t} & \frac{6}{7}e^{-2t}+\frac{1}{9}e^{-4t}-\frac{61}{63}e^{5t} \\ 0 & e^{-2t} & -\frac{3}{2}e^{-2t}+\frac{3}{2}e^{-4t} \\ 0 & 0 & e^{-4t} \end{bmatrix}$$

4. $$\begin{bmatrix} e^{3t} & -3e^{2t}+3e^{3t} & 15e^{2t}-15e^{3t} & \frac{3}{2}e^{2t}-15e^{3t}+\frac{27}{2}e^{4t} \\ 0 & e^{2t} & -5e^{2t}+5e^{3t} & -\frac{1}{2}e^{2t}+5e^{3t}-\frac{9}{2}e^{4t} \\ 0 & 0 & e^{3t} & e^{3t}-e^{4t} \\ 0 & 0 & 0 & e^{4t} \end{bmatrix}$$

5. $$\begin{bmatrix} -e^{3t}+2e^{-5t} & e^{3t}-e^{-5t} & e^{3t}-e^{-5t} \\ -e^{3t}+2e^{-5t}-e^{7t} & e^{3t}-e^{-5t}+e^{7t} & e^{3t}-e^{-5t} \\ -e^{3t}+e^{7t} & e^{3t}-e^{7t} & e^{3t} \end{bmatrix}$$

6. $$\begin{bmatrix} \frac{36}{5}e^{2t}-\frac{22}{35}e^{-3t}-\frac{39}{7}e^{4t} & \frac{2}{7}e^{-3t}-\frac{2}{7}e^{4t} & -\frac{36}{5}e^{2t}+\frac{12}{35}e^{-3t}+\frac{48}{7}e^{4t} \\ \frac{42}{5}e^{2t}-\frac{99}{35}e^{-3t}-\frac{39}{7}e^{4t} & \frac{9}{7}e^{-3t}-\frac{2}{7}e^{4t} & -\frac{42}{5}e^{2t}+\frac{54}{35}e^{-3t}+\frac{48}{7}e^{4t} \\ \frac{31}{5}e^{2t}-\frac{22}{35}e^{-3t}-\frac{39}{7}e^{4t} & \frac{2}{7}e^{-3t}-\frac{2}{7}e^{4t} & -\frac{31}{5}e^{2t}+\frac{12}{35}e^{-3t}+\frac{48}{7}e^{4t} \end{bmatrix}$$

7. $$\begin{bmatrix} 3e^{-t}-2e^{-3t} & 2e^{-t}-2e^{-3t} & 2e^{-t}-2e^{-3t} \\ 3e^{-3t}-3e^{4t} & 3e^{-3t}-2e^{4t} & 3e^{-3t}-3e^{4t} \\ -3e^{-t}+3e^{4t} & -2e^{-t}+2e^{4t} & -2e^{-t}+3e^{4t} \end{bmatrix}$$

8. $$\begin{bmatrix} 6e^{4t}-5e^{7t} & 2e^{4t}-2e^{7t} & -4e^{4t}+4e^{7t} \\ -5e^{4t}+5e^{7t} & -e^{4t}+2e^{7t} & 4e^{4t}-4e^{7t} \\ 5e^{4t}-5e^{7t} & 2e^{4t}-2e^{7t} & -3e^{4t}+4e^{7t} \end{bmatrix}$$

9. $$\begin{bmatrix} 2e^{-2t}-e^{9t} & -e^{-2t}+e^{9t} & -e^{-2t}+e^{9t} \\ 4e^{-2t}-4e^{9t} & -3e^{-2t}+4e^{9t} & -4e^{-2t}+4e^{9t} \\ -2e^{-2t}+2e^{9t} & 2e^{-2t}-2e^{9t} & 3e^{-2t}-2e^{9t} \end{bmatrix}$$

10. $$\begin{bmatrix} \frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} & 0 & 0 & -\frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} \\ 0 & \frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} & -\frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} & 0 \\ 0 & -\frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} & \frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} & 0 \\ -\frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} & 0 & 0 & \frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} \end{bmatrix}$$

11. $$\begin{bmatrix} e^{5t} & 9e^{2t}-9e^{3t} & 3e^{3t}-3e^{5t} & 9e^{2t}-9e^{3t} \\ -e^{-3t}+e^{5t} & -e^{2t}+2e^{-3t} & 3e^{-3t}-3e^{5t} & -e^{2t}+e^{-3t} \\ 0 & 3e^{2t}-3e^{3t} & e^{3t} & 3e^{2t}-3e^{3t} \\ e^{-3t}-e^{5t} & 2e^{2t}-2e^{-3t} & -3e^{-3t}+3e^{5t} & 2e^{2t}-e^{-3t} \end{bmatrix}$$

12. $$\begin{bmatrix} -12e^{-2t}+8e^{3t}+5e^{4t} & -15e^{-2t}+10e^{3t}+5e^{4t} & 15e^{-2t}-10e^{3t}-5e^{4t} & 6e^{-2t}-6e^{3t} \\ 6e^{-2t}-4e^{3t}-2e^{4t} & 8e^{-2t}-5e^{3t}-2e^{4t} & -7e^{-2t}+5e^{3t}+2e^{4t} & -3e^{-2t}+3e^{3t} \\ -6e^{-2t}+4e^{3t}+2e^{4t} & -7e^{-2t}+5e^{3t}+2e^{4t} & 8e^{-2t}-5e^{3t}-2e^{4t} & 3e^{-2t}-3e^{3t} \\ 4e^{-2t}-4e^{3t} & 5e^{-2t}-5e^{3t} & -5e^{-2t}+5e^{3t} & -2e^{-2t}+3e^{3t} \end{bmatrix}$$

13. $$\begin{bmatrix} -3e^{2t}+4e^{-5t} & -6e^{2t}+6e^{-5t} & 2e^{2t}-2e^{-5t} & 6e^{2t}-6e^{-5t} \\ 2e^{2t}-2e^{-5t} & 4e^{2t}-3e^{-5t} & -e^{2t}+e^{-5t} & -3e^{2t}+3e^{-5t} \\ -6e^{2t}+6e^{-5t} & -9e^{2t}+9e^{-5t} & 4e^{2t}-3e^{-5t} & 9e^{2t}-9e^{-5t} \\ 2e^{2t}-2e^{-5t} & 3e^{2t}-3e^{-5t} & -e^{2t}+e^{-5t} & -2e^{2t}+3e^{-5t} \end{bmatrix}$$

14. $$\begin{bmatrix} \frac{1}{2}e^{4t}+\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}e^{4t}-\frac{1}{2} \\ 0 & \frac{1}{2}e^{4t}+\frac{1}{2} & 0 & \frac{1}{2}e^{4t}-\frac{1}{2} & 0 \\ 0 & 0 & e^{2t} & 0 & 0 \\ 0 & \frac{1}{2}e^{4t}-\frac{1}{2} & 0 & \frac{1}{2}e^{4t}+\frac{1}{2} & 0 \\ \frac{1}{2}e^{4t}-\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}e^{4t}+\frac{1}{2} \end{bmatrix}$$

15. $$\begin{bmatrix} \frac{1}{2}e^{-6t}+\frac{1}{2}e^{6t} & -\frac{3}{4}e^{-6t}+\frac{3}{4}e^{6t} \\ -\frac{1}{3}e^{-6t}+\frac{1}{3}e^{6t} & \frac{1}{2}e^{-6t}+\frac{1}{2}e^{6t} \end{bmatrix}$$

16. $$\begin{bmatrix} e^{3t} & -3e^{-2t}+3e^{3t} & \frac{6}{5}e^{-2t}-\frac{6}{5}e^{3t} \\ 0 & e^{-2t} & -\frac{2}{5}e^{-2t}+\frac{2}{5}e^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix}$$

17. $$\begin{bmatrix} e^{-7t} & 0 & 0 \\ -\frac{1}{3}e^{2t}+\frac{1}{3}e^{-7t} & e^{2t} & 0 \\ \frac{2}{3}e^{2t}-\frac{2}{3}e^{-7t} & -2e^{2t}+2e^{-7t} & e^{-7t} \end{bmatrix}$$

18. $$\begin{bmatrix} e^{-2t} & 2e^{-2t}-2e^{3t} & -\frac{14}{5}e^{-2t}+\frac{14}{5}e^{3t} & -\frac{4}{5}e^{-2t}+\frac{4}{5}e^{3t} \\ 0 & e^{3t} & \frac{7}{5}e^{-2t}-\frac{7}{5}e^{3t} & -\frac{21}{5}e^{-2t}+\frac{21}{5}e^{3t} \\ 0 & 0 & e^{-2t} & -3e^{-2t}+3e^{3t} \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}$$

19. $$\begin{bmatrix} e^{-2t} & 4e^{-2t}-4e^{3t} & -\frac{8}{5}e^{-2t}+\frac{8}{5}e^{3t} & \frac{16}{5}e^{-2t}-\frac{16}{5}e^{3t} \\ 0 & e^{3t} & \frac{2}{5}e^{-2t}-\frac{2}{5}e^{3t} & -\frac{4}{5}e^{-2t}+\frac{4}{5}e^{3t} \\ 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{bmatrix}$$

20. $$\begin{bmatrix} e^{t} & -2e^{t}+2e^{3t} & 6e^{t}-6e^{3t} & -18e^{t}+18e^{3t} & -45e^{t}+45e^{3t} \\ 0 & e^{3t} & 3e^{t}-3e^{3t} & -9e^{t}+9e^{3t} & -\frac{45}{2}e^{t}+\frac{45}{2}e^{3t} \\ 0 & 0 & e^{t} & -3e^{t}+3e^{3t} & -\frac{15}{2}e^{t}+\frac{15}{2}e^{3t} \\ 0 & 0 & 0 & e^{3t} & -\frac{5}{2}e^{t}+\frac{5}{2}e^{3t} \\ 0 & 0 & 0 & 0 & e^{t} \end{bmatrix}$$

6.5 Exercises

1. a. $\langle\vec{v}\rangle_B = \langle -3, 7, -10\rangle$ and $\langle\vec{v}\rangle_{B'} = \langle 3, 8/3, 4/3\rangle$. b. The rrefs contained I_3 on the left side.

   c. $$C_{B,B'} = \begin{bmatrix} -2 & 1 & 1 \\ \frac{4}{3} & 0 & -\frac{2}{3} \\ -\frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

2. a. $\langle\vec{v}\rangle_B = \langle 4, -3, 6, 33/2\rangle$ and $\langle\vec{v}\rangle_{B'} = \langle 5, 3, -7/2, 15/2\rangle$. b. The rrefs contained I_4 on the left side.

   c. $$C_{B,B'} = \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & -1 \\ 0 & 2 & -6 & 3 \end{bmatrix}.$$

3. a. $T(\vec{v}) = -\frac{49}{2}(\langle 0, -1, 1\rangle) + 15(\langle 1, -1, 1\rangle) - 23(\langle 1, 2, 1\rangle) = \langle -8, -73/2, -65/2\rangle$

   b. $$[T] = \begin{bmatrix} -1 & 4 & -3 & 3 \\ -\frac{5}{2} & -\frac{7}{2} & \frac{13}{2} & 5 \\ -\frac{1}{2} & \frac{7}{2} & -\frac{1}{2} & 10 \end{bmatrix}$$

4. a. $\begin{bmatrix} 4 & 3 & 1 \\ -3 & 1 & 0 \\ -5 & -2 & 4 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} -3 \\ 7 \\ -10 \end{bmatrix} = \begin{bmatrix} -1 \\ 16 \\ -39 \\ 13 \end{bmatrix}$. Decoding:

$T(\vec{v}) = -1\langle 1,0,1,2\rangle + 16\langle 0,1,1,-1\rangle - 39\langle 0,0,2,1\rangle + 13\langle 0,0,0,-1\rangle = \langle -1,16,-63,-70\rangle$.

b. $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ -3 & 1 & 0 \\ -5 & -2 & 4 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 2 & -1 \\ -1 & -2 & 2 \\ -9 & 9 & 0 \\ 1 & 13 & -5 \end{bmatrix}$

5. a. $\begin{bmatrix} 6 & -3 & -1 \\ -2 & 1 & 0 \\ -7 & 2 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ 7 \\ -10 \end{bmatrix} = \begin{bmatrix} -29 \\ 13 \\ -5 \end{bmatrix}$. Decoding, we get:

$T(\vec{v}) = -29\langle 1,0,-1\rangle + 13\langle 1,1,2\rangle - 5\langle 0,1,1\rangle = \langle -16,8,50\rangle$.

b. $[T] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 6 & -3 & -1 \\ -2 & 1 & 0 \\ -7 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & -\frac{5}{2} \\ -5 & 0 & 4 \\ -\frac{15}{2} & -\frac{9}{2} & \frac{19}{2} \end{bmatrix}$

6. a. $\begin{bmatrix} 7 & 3 & 1 \\ -1 & -4 & 0 \\ 3 & 5 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 26 \\ 8 \\ -17 \end{bmatrix}$ b. $\begin{bmatrix} 0 & \frac{3}{2} & -\frac{3}{2} \\ 6 & \frac{21}{2} & \frac{11}{2} \\ -7 & -\frac{31}{2} & -\frac{19}{2} \end{bmatrix}$

7. a. $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\} = \{\langle -3,1,6,-5\rangle, \langle 4,2,-4,-4\rangle, \langle 1,4,7,3\rangle\}$
   b. $B' = \{\langle 1,0,0,7\rangle, \langle 0,1,0,-8\rangle, \langle 0,0,1,4\rangle\}$
   c. $\langle 18,4,-24,-2\rangle = -2\langle -3,1,6,-5\rangle + 3\langle 4,2,-4,-4\rangle$
   d. $\langle 18,4,-24,-2\rangle = 18\langle 1,0,0,7\rangle + 4\langle 0,1,0,-8\rangle - 24\langle 0,0,1,4\rangle$
   e. $C_{B,B'} = \begin{bmatrix} -3 & 4 & 18 \\ 1 & 2 & 4 \\ 6 & -4 & -24 \end{bmatrix}$
   f. $\begin{bmatrix} -3 & 4 & 18 \\ 1 & 2 & 4 \\ 6 & -4 & -24 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 18 \\ 4 \\ -24 \end{bmatrix}$

8. a. $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{\langle -3,1,6,-5\rangle, \langle 4,2,-4,-4\rangle, \langle 1,4,7,3\rangle\}$
   b. $B' = \{\langle 1,0,0,7\rangle, \langle 0,1,0,-8\rangle, \langle 0,0,1,4\rangle\}$
   c. $\langle -10,-3,1,-42\rangle = 5\langle -3,1,6,-5\rangle + 2\langle 4,2,-4,-4\rangle - 3\langle 1,4,7,3\rangle$
   d. $\langle -10,-3,1,-42\rangle = -10\langle 1,0,0,7\rangle - 3\langle 0,1,0,-8\rangle + 1\langle 0,0,1,4\rangle$

e. $C_{B,B'} = \begin{bmatrix} -3 & 4 & 1 \\ 1 & 2 & 4 \\ 6 & -4 & 7 \end{bmatrix}$

f. $\begin{bmatrix} -3 & 4 & 1 \\ 1 & 2 & 4 \\ 6 & -4 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -10 \\ -3 \\ 1 \end{bmatrix}$

9. a. $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{\langle -3, 12, 5, 2, -2\rangle, \langle 1, -4, 4, 3, -4\rangle, \langle 4, -16, -6, -4, 18\rangle\}$

b. $B' = \{\langle 1, -4, 0, 0, 3\rangle, \langle 0, 0, 1, 0, 5\rangle, \langle 0, 0, 0, 1, -9\rangle\}$

For (c) and (d), there are no vectors from S which are not in B.

e. $C_{B,B'} = \begin{bmatrix} -3 & 1 & 4 \\ 5 & 4 & -6 \\ 2 & 3 & -4 \end{bmatrix}$

10. a. $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{\langle -3, -4, -2, 9, 1, 1\rangle, \langle 1, 2, 4, 9, 11, -11\rangle, \langle 4, 3, 5, 16, 1, 8\rangle\}$

b. $B' = \{\langle 1, 0, 0, 3, -5, 9\rangle, \langle 0, 1, 0, -7, 2, -6\rangle, \langle 0, 0, 1, 5, 3, -2\rangle\}$

c.

$\langle -21, -36, -26, 59, -45, 79\rangle = 8\langle -3, -4, -2, 9, 1, 1\rangle - 5\langle 1, 2, 4, 9, 11, -11\rangle + 2\langle 4, 3, 5, 16, 1, 8\rangle$

$\langle -20, -37, -23, 84, -43, 88\rangle = 9\langle -3, -4, -2, 9, 1, 1\rangle - 5\langle 1, 2, 4, 9, 11, -11\rangle + 3\langle 4, 3, 5, 16, 1, 8\rangle$

d.

$\langle -21, -36, -26, 59, -45, 79\rangle = -21\langle 1, 0, 0, 3, -5, 9\rangle - 36\langle 0, 1, 0, -7, 2, -6\rangle - 26\langle 0, 0, 1, 5, 3, -2\rangle$

$\langle -20, -37, -23, 84, -43, 88\rangle = -20\langle 1, 0, 0, 3, -5, 9\rangle - 37\langle 0, 1, 0, -7, 2, -6\rangle - 23\langle 0, 0, 1, 5, 3, -2$

e. $C_{B,B'} = \begin{bmatrix} -3 & 1 & 4 \\ -4 & 2 & 3 \\ -2 & 4 & 5 \end{bmatrix}$

f. $\begin{bmatrix} -3 & 1 & 4 \\ -4 & 2 & 3 \\ -2 & 4 & 5 \end{bmatrix}\begin{bmatrix} 8 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} -21 \\ -36 \\ -26 \end{bmatrix}$

$\begin{bmatrix} -3 & 1 & 4 \\ -4 & 2 & 3 \\ -2 & 4 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} -20 \\ -37 \\ -23 \end{bmatrix}$

11. a. $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5\} =$

$\{\langle -5, 3, -3, 2, -14, -4\rangle, \langle 3, -4, -7, -5, -21, 7\rangle, \langle 2, -1, 2, 0, 11, 2\rangle, \langle -1, 2, 5, 3, 17, -8\rangle\}$

b. $B' = \{\langle 1, 0, 3, 0, 7, 0\rangle, \langle 0, 1, 4, 0, 3, 0\rangle, \langle 0, 0, 0, 1, 6, 0\rangle, \langle 0, 0, 0, 0, 0, 1\rangle\}$

c. $\langle -21, 17, 5, 16, 0, -26\rangle = 3\langle -5, 3, -3, 2, -14, -4\rangle - 2\langle 3, -4, -7, -5, -21, 7\rangle$

d. $\langle -21, 17, 5, 16, 0, -26\rangle = -21\langle 1, 0, 3, 0, 7, 0\rangle + 17\langle 0, 1, 4, 0, 3, 0\rangle$
$+ 16\langle 0, 0, 0, 1, 6, 0\rangle - 26\langle 0, 0, 0, 0, 0, 1\rangle$

e. $C_{B,B'} = \begin{bmatrix} -5 & 3 & 2 & -1 \\ 3 & -4 & -1 & 2 \\ 2 & -5 & 0 & 3 \\ -4 & 7 & 2 & -8 \end{bmatrix}$

f. $\begin{bmatrix} -5 & 3 & 2 & -1 \\ 3 & -4 & -1 & 2 \\ 2 & -5 & 0 & 3 \\ -4 & 7 & 2 & -8 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -21 \\ 17 \\ 16 \\ -26 \end{bmatrix}$

12. a. $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5\} =$
$\{\langle -4,-5,-1,3,7,1\rangle, \langle 2,3,-1,-1,-8,9\rangle, \langle -1,0,-4,2,2,1\rangle, \langle 3,2,6,-5,-4,-12\rangle\}$
b. $B' = \{\langle 1,0,4,0,0,9\rangle, \langle 0,1,-3,0,0,-6\rangle, \langle 0,0,0,1,0,7\rangle, \langle 0,0,0,0,1,-2\rangle\}$
c. $\langle -2,-1,-5,3,-10,29\rangle = 2\langle -4,-5,-1,3,7,1\rangle + 3\langle 2,3,-1,-1,-8,9\rangle$
d. $\langle -2,-1,-5,3,-10,29\rangle = -2\langle 1,0,4,0,0,9\rangle - \langle 0,1,-3,0,0,-6\rangle$
$+ 3\langle 0,0,0,1,0,7\rangle - 10\langle 0,0,0,0,1,-2\rangle$

e. $C_{B,B'} = \begin{bmatrix} -4 & 2 & -1 & 3 \\ -5 & 3 & 0 & 2 \\ 3 & -1 & 2 & -5 \\ 7 & -8 & 2 & -4 \end{bmatrix}$

f. $\begin{bmatrix} -4 & 2 & -1 & 3 \\ -5 & 3 & 0 & 2 \\ 3 & -1 & 2 & -5 \\ 7 & -8 & 2 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \\ -10 \end{bmatrix}$

13. a. $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{\langle -3,1,4,-21,-20\rangle, \langle -4,2,3,-36,-37\rangle, \langle -2,4,5,-26,-23\rangle\}$
b. $B' = \{\langle 1,0,0,8,9\rangle, \langle 0,1,0,-5,-5\rangle, \langle 0,0,1,2,3\rangle\}$
c. $\langle 9,9,16,59,84\rangle = 3\langle -3,1,4,-21,-20\rangle - 7\langle -4,2,3,-36,-37\rangle + 5\langle -2,4,5,-26,-23\rangle$
$\langle 1,11,1,-45,-43\rangle = -5\langle -3,1,4,-21,-20\rangle + 2\langle -4,2,3,-36,-37\rangle + 3\langle -2,4,5,-26,-23\rangle$
$\langle 1,-11,8,79,88\rangle = 9\langle -3,1,4,-21,-20\rangle - 6\langle -4,2,3,-36,-37\rangle - 2\langle -2,4,5,-26,-23\rangle$
d. $\langle 9,9,16,59,84\rangle = 9\langle 1,0,0,8,9\rangle + 9\langle 0,1,0,-5,-5\rangle + 16\langle 0,0,1,2,3\rangle$
$\langle 1,11,1,-45,-43\rangle = \langle 1,0,0,8,9\rangle + 11\langle 0,1,0,-5,-5\rangle + \langle 0,0,1,2,3\rangle$
$\langle 1,-11,8,79,88\rangle = \langle 1,0,0,8,9\rangle - 11\langle 0,1,0,-5,-5\rangle + 8\langle 0,0,1,2,3\rangle$

e. $C_{B,B'} = \begin{bmatrix} -3 & -4 & -2 \\ 1 & 2 & 4 \\ 4 & 3 & 5 \end{bmatrix}$

f. $\begin{bmatrix} -3 & -4 & -2 \\ 1 & 2 & 4 \\ 4 & 3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ -7 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 16 \end{bmatrix}$

$$\begin{bmatrix} -3 & -4 & -2 \\ 1 & 2 & 4 \\ 4 & 3 & 5 \end{bmatrix}\begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -4 & -2 \\ 1 & 2 & 4 \\ 4 & 3 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ -6 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \\ 8 \end{bmatrix}$$

14. a. $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \{\langle -4,-5,3,19,2,-8\rangle, \langle -8,-1,2,-28,3,-26\rangle, \langle 2,2,-1,-5,0,15\rangle, \langle 7,3,-4,5,-5,-8\rangle\}$

b. $B' = \{\langle 1,0,0,5,0,8\rangle, \langle 0,1,0,-6,0,3\rangle, \langle 0,0,1,3,0,7\rangle, \langle 0,0,0,0,1,9\rangle\}$

c. $\langle 8,7,-10,-32,-8,-57\rangle = 5\langle -4,-5,3,19,2,-8\rangle + 4\langle -8,-1,2,-28,3,-26\rangle + 9\langle 2,2,-1,-5,0,15\rangle + 6\langle 7,3,-4,5,-5,-8\rangle$

d. $\langle 8,7,-10,-32,-8,-57\rangle = 8\langle 1,0,0,5,0,8\rangle + 7\langle 0,1,0,-6,0,3\rangle - 10\langle 0,0,1,3,0,7\rangle - 8\langle 0,0,0,0,1,9\rangle$

e. $C_{B,B'} = \begin{bmatrix} -4 & -8 & 2 & 7 \\ -5 & -1 & 2 & 3 \\ 3 & 2 & -1 & -4 \\ 2 & 3 & 0 & -5 \end{bmatrix}$

f. $\begin{bmatrix} -4 & -8 & 2 & 7 \\ -5 & -1 & 2 & 3 \\ 3 & 2 & -1 & -4 \\ 2 & 3 & 0 & -5 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ -10 \\ -8 \end{bmatrix}$

6.6 Exercises

1. a. $\langle \vec{v}\rangle_B = \langle -11,6,9\rangle$ and $\langle \vec{v}\rangle_{B'} = \langle 3,0,2\rangle$. c. $C_{B,B'} = \begin{bmatrix} -1 & -\frac{4}{3} & 0 \\ 1 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$.

2. a. $\langle \vec{v}\rangle_B = \langle 2,4,1,-5\rangle$ and $\langle \vec{v}\rangle_{B'} = \langle 5,-3,16,-70\rangle$. c. $C_{B,B'} = \begin{bmatrix} 1 & 2 & 5 & 2 \\ 1 & -1 & -1 & 0 \\ 4 & 6 & 14 & 6 \\ -19 & -27 & -64 & -28 \end{bmatrix}$.

3. a. $T(\vec{v}) = 27(x - x^2) - 49(1 + x) + 38(2 - x^2) = 27 - 22x - 65x^2$

b. $[T]_{S,S'} = \begin{bmatrix} 2 & 17 & 26 & 18 \\ \frac{7}{2} & \frac{29}{2} & \frac{19}{2} & 17 \\ \frac{3}{2} & \frac{11}{2} & -\frac{15}{2} & 13 \end{bmatrix}$.

4. a. $T(\vec{v}) = 27 + 36x - 169x^2 + 144x^3$. b. $[T]_{S,S'} = \begin{bmatrix} 2 & 3 & -2 \\ 3 & -2 & -6 \\ -14 & -6 & 19 \\ 10 & 8 & -16 \end{bmatrix}$

5. a. $T(\vec{v}) = -21 + 29x - 29x^2$. b. $[B]_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $[B]_S^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$.

c. $[T]_S = \begin{bmatrix} -3 & 1 & 4 \\ 2 & 2 & -3 \\ -3 & 2 & 3 \end{bmatrix}$. e. $det(T) = -7$. f. Yes. $[T^{-1}]_B = \begin{bmatrix} \frac{5}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{3}{7} & -\frac{2}{7} & -\frac{3}{7} \\ -\frac{1}{7} & -\frac{3}{7} & -\frac{1}{7} \end{bmatrix}$.

6. a. $T(\vec{v}) = 116 - 63x + 27x^2 + 19x^3$.

b. $[B]_S = \begin{bmatrix} 1 & 2 & 5 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$ and $[B]_S^{-1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 \\ \frac{1}{2} & \frac{5}{2} & \frac{3}{2} & 3 \end{bmatrix}$.

c. $[T]_S = \begin{bmatrix} -\frac{13}{2} & -\frac{99}{2} & -\frac{51}{2} & -51 \\ \frac{9}{2} & \frac{57}{2} & \frac{33}{2} & 33 \\ -2 & -11 & -5 & -12 \\ -\frac{3}{2} & -\frac{19}{2} & -\frac{13}{2} & -12 \end{bmatrix}$ d. $det(T) = 0$. e. No.

7. a. $[D]_B = \begin{bmatrix} 2 & 3 & -1 \\ -2 & -3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ b. $[D]_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ c. $det(T) = 0$. d. No.

8. a. $[D]_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ -4 & \frac{9}{2} & -\frac{1}{2} & 0 \end{bmatrix}$ b. $[D]_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ c. $det(T) = 0$. d. No.

9. a. The members of B' are non-zero, non-parallel linear combinations of $\sin(x)$ and $\cos(x)$.

b. $\begin{bmatrix} \sqrt{3} & -1 \\ -1 & \sqrt{3} \end{bmatrix}$ c. $[T]_B = \begin{bmatrix} 5 + 3\sqrt{3} & -4\sqrt{3} - 6 \\ 4\sqrt{3} + 6 & -11 - 3\sqrt{3} \end{bmatrix}$ d. $det(T) = 2$.

e. Yes; $[T^{-1}]_B = \begin{bmatrix} -\frac{11}{2} - \frac{3}{2}\sqrt{3} & 2\sqrt{3} + 3 \\ -2\sqrt{3} - 3 & \frac{5}{2} + \frac{3}{2}\sqrt{3} \end{bmatrix}$ f. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

g. $det(D) = 1$. h. Yes. $[D^{-1}]_B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

10. a. $[D]_B = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$. b. $det(D) = -8$ c. $[D^{-1}]_B = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$.

6.7 Exercises

1. a. $[T]_S = \begin{bmatrix} 0 & -5 & -14 \\ 0 & 3 & -10 \\ 0 & 0 & 14 \end{bmatrix}$ b. $det(T) = 0$; c. $p(\lambda) = \lambda(\lambda - 3)(\lambda - 14)$ d. $\lambda = 0, 3, 14$
   e. $Eig(T, 0) = Span(\{1\})$; $Eig(T, 3) = Span(\{-5 + 3x\})$;
   $Eig(T, 14) = Span(\{52 + 70x - 77x^2\})$
   f. $[T]_B = Diag(0, 3, 14)$, where $B = \{1, -5 + 3x, 52 + 70x - 77x^2\}$.
2. a. $[T]_S = \begin{bmatrix} 4 & 5 & -8 & 0 \\ 0 & 6 & 14 & -24 \\ 0 & 0 & 14 & 27 \\ 0 & 0 & 0 & 28 \end{bmatrix}$ b. $det(T) = 9408$; c.
   $p(\lambda) = (\lambda - 4)(\lambda - 6)(\lambda - 14)(\lambda - 28)$
   d. $\lambda = 4, 6, 14, 28$
   e. $Eig(T, 4) = Span(\{1\})$; $Eig(T, 6) = Span(\{5 + 2x\})$;
   $Eig(T, 14) = Span(\{3 + 70x + 40x^2\})$;
   $Eig(T, 28) = Span(\{-757 + 168x + 2376x^2 + 1232x^3\})$;
   f. $[T]_B = Diag(4, 6, 14, 28)$, where
   $B = \{1, 5 + 2x, 3 + 70x + 40x^2, -757 + 168x + 2376x^2 + 1232x^3\}$.
3. a. $[T]_S = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$ b. $det(T) = 25$; c. $p(\lambda) = \lambda^2 + 25$
   d. The eigenvalues are imaginary, so . . . e. there are no eigenvectors for T, and consequently, . . .
   f. T is not diagonalizable.
4. a. $[D]_S = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ b. $det(D) = -10$; c. $p(\lambda) = (\lambda + 1)(\lambda - 2)(\lambda - 5)$
   d. $\lambda = -1, 2, 5$; f. $Eig(D, -1) = Span(\{e^{-x}\})$; $Eig(D, 2) = Span(\{e^{2x}\})$;
   $Eig(D, 5) = Span(\{e^{5x}\})$;
   g. $[D]_S$ is already diagonal, so it is diagonalizable.

5. a. $[D]_S = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ b. $det(D) = 27$; c. $p(\lambda) = (\lambda - 3)^3$ d. $\lambda = 3$
   e. $Eig(D,3) = Span(\{e^{3x}\})$ f. D is not diagonalizable.
6. a. $\lambda = -1$ for both $\sin(x)$ and $\cos(x)$. b. The eigenvalue of $e^{\lambda x}$ is λ^2. c. $e^{\sqrt{\mu}\,x}$ d. $-\lambda^2$
   e. It has the same eigenvalue, $-\lambda^2$. f. The common eigenvalue is 1.
7. $[T]_B = Diag(5,-1,4)$, where $B = \{1, 1-3x, 3+6x+5x^2\}$
8. $[T]_S = \begin{bmatrix} 3 & 2 & 1 \\ -5 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{12} & -\frac{1}{4} & \frac{1}{6} \\ \frac{5}{12} & \frac{1}{4} & \frac{1}{6} \\ \frac{5}{12} & \frac{1}{4} & -\frac{5}{6} \end{bmatrix} = \begin{bmatrix} -\frac{67}{12} & -\frac{23}{4} & -\frac{17}{6} \\ \frac{5}{12} & \frac{17}{4} & -\frac{5}{6} \\ -\frac{25}{6} & -\frac{5}{2} & \frac{4}{3} \end{bmatrix}$
9. There are 366 equivalence classes, including February 29.

Chapter Seven Exercises

7.1 Exercises

9. -46 10. $-22/5$ 11. 16 12. -276 13. 22 14. -72 15. 38 16. $-10,892$ 17. $1/2$ 18. 0

19. $r(x) = (x+1)(x-1)(x-2)(x-4)$ or any scalar multiple thereof.

20. No. 21. No. 22. Yes. 23. $-\pi/4$ 30. b. removable discontinuity

31. a. Further hint: since the series $\sum a_n$ converges, the terms a_n must converge to 0, so therefore if n is large enough, $|a_n| < 1$. c. (use geometric series formula) 1/5; d. $-1/16$.

7.2 Exercises

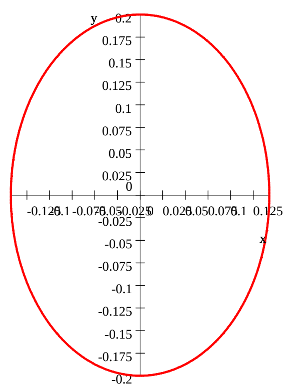
1. $\sqrt{279}$; $\pm\vec{u}/\sqrt{279}$ 2. $\sqrt{341}$; $\pm\vec{u}/\sqrt{341}$ 3. $\sqrt{131}$; $\pm\vec{u}/\sqrt{131}$ 4. $\sqrt{354}$; $\pm p(x)/\sqrt{354}$ 5. $\sqrt{1802}$; $\pm p(x)/\sqrt{1802}$

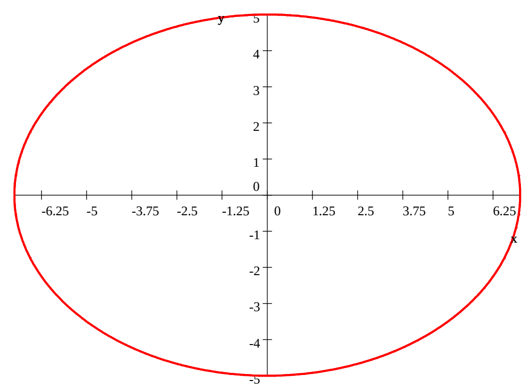
6. $\theta = \cos^{-1}\left(312/\sqrt{8051}\right)$ and $d(\vec{u},\vec{v}) = \sqrt{8}$ 7. $\theta = \cos^{-1}\left(16/\sqrt{17510}\right)$ and $d(\vec{u},\vec{v}) = \sqrt{241}$

8. $\theta = \cos^{-1}(11/15)$ and $d(\vec{u},\vec{v}) = \sqrt{65}$ 9. $\theta = \cos^{-1}\left(-298/\sqrt{131,334}\right)$ and $d(\vec{u},\vec{v}) = \sqrt{1321}$

10. $\theta = \cos^{-1}\left(-10892/\sqrt{120816920}\right)$ and $d(\vec{u},\vec{v}) = \sqrt{47290}$

11. $8/\sqrt{15}$; 12. $\cos(\theta) = 2/\sqrt{\pi^2-4}$, so $\theta \approx 0.6$ radians; 13. $49x^2 + 25y^2 = 1$ is an ellipse (left, below):





14. $\dfrac{x^2}{49} + \dfrac{y^2}{25} = 1$ is an ellipse (above, right).

15. $4x^2 + y^2 + 25z^2 = 1$ is an ellipsoid with vertices $(\pm 1/2, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1/5)$

16. $\sqrt{7342}$

17. No. $19 > 15$, so the conditions violate the Cauchy-Schwarz Inequality. 18. $\|\vec{u}\| = 13$ and $\|\vec{v}\| = 5$.

37. You get an isosceles triangle.

7.3 Exercises

1. $\left\{ \frac{1}{\sqrt{3}}\langle 1,1,-1\rangle, \frac{1}{\sqrt{6}}\langle 2,-1,1\rangle, \frac{1}{\sqrt{2}}\langle 0,1,1\rangle \right\}$

2. $\left\{\frac{1}{\sqrt{2}}\langle 1,0,1\rangle, \frac{1}{\sqrt{3}}\langle -1,1,1\rangle, \frac{1}{\sqrt{6}}\langle 1,2,-1\rangle\right\}$

3. $\left\{\frac{1}{\sqrt{12}}\langle 1,1,-1\rangle, \frac{1}{\sqrt{24}}\langle 2,-1,1\rangle, \frac{1}{\sqrt{120}}\langle 0,3,5\rangle\right\}$; different answer.

4. $\left\{\frac{1}{\sqrt{7}}\langle 1,1,-1\rangle, \frac{1}{\sqrt{581}}\langle 6,-7,-8\rangle, \frac{1}{\sqrt{4980}}\langle 15,24,-20\rangle\right\}$; different answer.

5. $\left\{\frac{1}{\sqrt{5}}\langle 1,1,-1\rangle, \frac{1}{\sqrt{5}}\langle 2,-3,3\rangle, \langle 1,-1,2\rangle\right\}$; different answer.

6. $\left\{\frac{1}{\sqrt{2}}\langle 1,0,1\rangle, \frac{1}{\sqrt{11}}\langle 2,-1,0\rangle, \frac{1}{\sqrt{22}}\langle -5,8,-11\rangle\right\}$

7. $\left\{\frac{1}{2}\langle 1,-1,1,-1\rangle, \frac{1}{2\sqrt{11}}\langle 5,-1,-3,3\rangle, \frac{1}{\sqrt{330}}\langle 7,14,-2,-9\rangle, \frac{1}{\sqrt{30}}\langle 1,2,4,3\rangle\right\}$

8. $\left\{\frac{1}{\sqrt{3}}\langle 1,-1,0,1\rangle, \frac{1}{\sqrt{15}}\langle 1,2,-3,1\rangle, \frac{1}{3\sqrt{10}}\langle 7,4,4,-3\rangle, \frac{1}{3\sqrt{2}}\langle -1,2,2,3\rangle\right\}$

9. $\left\{\frac{1}{\sqrt{14}}\langle 1,-1,1,-1\rangle, \frac{1}{\sqrt{2198}}\langle 19,-5,-9,9\rangle, \frac{1}{\sqrt{125286}}\langle 33,264,-90,-67\rangle, \frac{1}{2\sqrt{399}}\langle 3,24,16\right.$

10. $\left\{\frac{1}{\sqrt{11}}\langle 1,-1,0,1\rangle, \frac{1}{\sqrt{473}}\langle 1,10,-11,1\rangle, \frac{1}{\sqrt{21930}}\langle 57,54,18,-29\rangle, \frac{1}{\sqrt{1020}}\langle -3,24,8,6\rangle\right\}$ different answer.

11. $\left\{\frac{1}{\sqrt{17}}x^2, \frac{1}{6\sqrt{17}}(7x^2+17x), \frac{1}{2}(x^2+x-2)\right\}$

12. $\left\{\frac{1}{\sqrt{30}}(x^2+1), \frac{1}{\sqrt{330}}(8x^2+15x-7), \frac{1}{\sqrt{99}}(4x^2+2x-9)\right\}$

13. $\left\{\sqrt{5}\,x^2, \sqrt{3}\,(5x^2-4x), 10x^2-12x+3\right\}$

14. $\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{42}}(3x+1), \frac{1}{\sqrt{126}}(7x^2+8x-9)\right\}$; different answer.

15. $\left\{\sqrt{7}\,x^3, \sqrt{5}\,(6x^2-7x^3), \sqrt{3}\,(21x^3-30x^2+10x), -35x^3+60x^2-30x+4\right\}$

16. $\langle\vec{u}\rangle_S = \left\langle -\sqrt{3}, 3\sqrt{6}/2, -3\sqrt{2}/2\right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle -2\sqrt{3}, -\sqrt{6}/2, 13\sqrt{2}/2\right\rangle$

17. $\langle\vec{u}\rangle_S = \left\langle \frac{3}{\sqrt{2}}, -\frac{5}{\sqrt{3}}, -\frac{7}{\sqrt{6}}\right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle \frac{5}{\sqrt{2}}, \frac{16}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right\rangle$

18. $\langle\vec{u}\rangle_S = \left\langle -\frac{15}{\sqrt{12}}, \frac{39}{\sqrt{24}}, -\frac{45}{\sqrt{120}}\right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle -\frac{11}{\sqrt{12}}, -\frac{25}{\sqrt{24}}, \frac{195}{\sqrt{120}}\right\rangle$

19. $\langle\vec{u}\rangle_S = \left\langle 11\sqrt{7}, \frac{3}{83}\sqrt{581}, -\frac{102}{83}\sqrt{1245}\right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle 12\sqrt{7}, -\frac{85}{83}\sqrt{581}, -\frac{98}{83}\sqrt{1245}\right\rangle$

20. $\langle\vec{u}\rangle_S = \left\langle \frac{1}{\sqrt{5}}, \frac{12}{\sqrt{5}}, -3\right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle -\frac{12}{\sqrt{5}}, -\frac{34}{\sqrt{5}}, 13\right\rangle$

21. $\langle\vec{u}\rangle_S = \left\langle -\frac{5}{\sqrt{2}}, \frac{16}{\sqrt{11}}, -\frac{7}{\sqrt{22}}\right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle \frac{15}{\sqrt{2}}, -\frac{59}{\sqrt{11}}, -\frac{1}{\sqrt{22}}\right\rangle$

22. $\langle\vec{u}\rangle_S = \left\langle -\frac{1}{2}, \frac{3}{2\sqrt{11}}, \frac{145}{\sqrt{330}}, \frac{-5}{\sqrt{30}}\right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle \frac{17}{2}, \frac{-3}{2\sqrt{11}}, \frac{20}{\sqrt{330}}, \frac{20}{\sqrt{30}}\right\rangle$

23. $\langle\vec{u}\rangle_S = \left\langle -\frac{7}{3}\sqrt{3}, \frac{17}{15}\sqrt{15}, \frac{49}{30}\sqrt{10}, -\frac{7}{6}\sqrt{2}\right\rangle$, and

$\langle\vec{v}\rangle_S = \left\langle \frac{4}{3}\sqrt{3}, -\frac{23}{15}\sqrt{15}, \frac{32}{15}\sqrt{10}, -\frac{2}{3}\sqrt{2} \right\rangle$

24. $\langle\vec{u}\rangle_S = \left\langle \frac{12}{7}\sqrt{14}, \frac{18}{1099}\sqrt{2198}, \frac{688}{20881}\sqrt{125286}, -\frac{5}{133}\sqrt{399} \right\rangle$, and
$\langle\vec{v}\rangle_S = \left\langle \frac{61}{14}\sqrt{14}, \frac{39}{2198}\sqrt{2198}, -\frac{92}{20881}\sqrt{125286}, \frac{40}{133}\sqrt{399} \right\rangle$

25. $\langle\vec{u}\rangle_S = \left\langle \frac{-18}{\sqrt{11}}, \frac{114}{\sqrt{473}}, \frac{1596}{\sqrt{21930}}, -\frac{84}{\sqrt{1020}} \right\rangle$, and
$\langle\vec{v}\rangle_S = \left\langle \frac{4}{\sqrt{11}}, -\frac{249}{\sqrt{473}}, \frac{1932}{\sqrt{21930}}, -\frac{48}{\sqrt{1020}} \right\rangle$

26. $\langle\vec{u}\rangle_S = \left\langle -56\sqrt{17}/17, -3\sqrt{17}/17, 5 \right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle 73\sqrt{17}/17, -48\sqrt{17}/17, 4 \right\rangle$

27. $\langle\vec{u}\rangle_S = \left\langle \frac{-78}{\sqrt{30}}, \frac{36}{\sqrt{330}}, \frac{6}{\sqrt{99}} \right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle \frac{82}{\sqrt{30}}, \frac{-184}{\sqrt{330}}, \frac{117}{\sqrt{99}} \right\rangle$

28. $\langle\vec{u}\rangle_S = \left\langle -\frac{41}{30}\sqrt{5}, \frac{3}{2}\sqrt{3}, -\frac{5}{3} \right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle -\frac{67}{30}\sqrt{5}, \frac{11}{6}\sqrt{3}, -\frac{4}{3} \right\rangle$

29. $\langle\vec{u}\rangle_S = \left\langle \frac{-22}{\sqrt{3}}, \frac{44}{\sqrt{42}}, \frac{-18}{\sqrt{126}} \right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle \frac{9}{\sqrt{3}}, \frac{-132}{\sqrt{42}}, \frac{54}{\sqrt{126}} \right\rangle$

30. $\langle\vec{u}\rangle_S = \left\langle \frac{461}{294}\sqrt{7}, \frac{52}{105}\sqrt{5}, \frac{29}{210}\sqrt{3}, -\frac{3}{35} \right\rangle$, and
$\langle\vec{v}\rangle_S = \left\langle \frac{433}{1470}\sqrt{7}, -\frac{1}{21}\sqrt{5}, -\frac{11}{210}\sqrt{3}, \frac{1}{7} \right\rangle$

31. a. $\int_0^{2\pi} \sin(x)\cos(x)dx = \int_0^{2\pi} \sin(2x)\cos(x)dx = \int_0^{2\pi} \sin(2x)\sin(x)dx = 0$,
$\int_0^{2\pi} \sin^2(2x)dx = \int_0^{2\pi} \sin^2(x)dx = \int_0^{2\pi} \cos^2(x)dx = \pi$

b. $\left\{ \frac{1}{\sqrt{\pi}}\sin(x), \frac{1}{\sqrt{\pi}}\cos(x), \frac{1}{\sqrt{\pi}}\sin(2x) \right\}$ c. $\langle\vec{u}\rangle_S = \sqrt{\pi}\langle 2, 7, -3\rangle$, and
$\langle\vec{v}\rangle_S = \sqrt{\pi}\langle 5, -2, 1\rangle$

32. B is linearly dependent, because $\vec{w}_3$ is in the Span of $\{\vec{w}_1, \vec{w}_2\}$.

34. $p(x) = (x+3)(x-1)$, $q(x) = (x+3)(x-4)$, $r(x) = (x-1)(x-4)$ is a possible answer.

7.4 Exercises

1. $\{\langle 1,-1,1\rangle\}$ 2. $\{\langle 1,1,0\rangle, \langle -3,0,1\rangle\}$ 3. $\{\langle 15,-12,20\rangle\}$ 4. $\{\langle 5,4,0\rangle, \langle -3,0,2\rangle\}$
5. $\{\langle 1,0,1,0\rangle, \langle -1,-2,0,1\rangle\}$ 6. $\{\langle 3,0,4,0\rangle, \langle -3,-24,0,2\rangle\}$ 7. $\{\langle -9,-24,-8,2\rangle\}$ 8. $\{\langle 2,3,-3\rangle\}$
9. $\{\langle 3,5,0\rangle, \langle 1,0,1\rangle\}$ 10. $\{7x^2+17x, -5x^2+17\}$ 11. $\{5x^2+7x-9\}$
12. $\{224+888x-251x^2, 414-3827x+251x^3\}$ 13. $\{1-5x, 8-5x^2, 7-5x^3\}$
14. $\{25x^2-17x, 50x^2-17\}$

15. for W : $\left\{ \frac{1}{\sqrt{3}}\langle 1,1,-1\rangle, \frac{1}{\sqrt{6}}\langle 2,-1,1\rangle \right\}$; for $W^\perp$: $\left\{ \frac{1}{\sqrt{2}}\langle 0,1,1\rangle \right\}$

16. for W : $\left\{ \frac{1}{\sqrt{2}}\langle 1,0,1\rangle \right\}$; for $W^\perp$: $\left\{ , \frac{1}{\sqrt{3}}\langle -1,1,1\rangle, \frac{1}{\sqrt{6}}\langle 1,2,-1\rangle \right\}$

17. for W : $\left\{ \frac{1}{2\sqrt{3}}\langle 1,1,-1\rangle \right\}$; for $W^\perp$: $\left\{ \frac{1}{\sqrt{24}}\langle 2,-1,1\rangle, \frac{1}{\sqrt{120}}\langle 0,3,5\rangle \right\}$

18. for W : $\left\{ \frac{1}{\sqrt{7}}\langle 1,1,-1\rangle, \frac{1}{\sqrt{581}}\langle 6,-7,-8\rangle \right\}$; for $W^\perp$: $\left\{ \frac{1}{\sqrt{4980}}\langle 15,24,-20\rangle \right\}$

19. for W : $\left\{\frac{1}{\sqrt{2}}\langle 1,0,1\rangle, \frac{1}{\sqrt{11}}\langle 2,-1,0\rangle\right\}$; for $W^\perp$: $\left\{\frac{1}{\sqrt{22}}\langle -5,8,-11\rangle\right\}$

20. for W : $\left\{\frac{1}{2}\langle 1,-1,1,-1\rangle, \frac{1}{2\sqrt{11}}\langle 5,-1,-3,3\rangle\right\}$; for
$W^\perp$: $\left\{\frac{1}{\sqrt{330}}\langle 7,14,-2,-9\rangle, \frac{1}{\sqrt{30}}\langle 1,2,4,3\rangle\right\}$

21. for W : $\left\{\frac{1}{\sqrt{3}}\langle 1,-1,0,1\rangle\right\}$; for
$W^\perp$: $\left\{\frac{1}{\sqrt{15}}\langle 1,2,-3,1\rangle, \frac{1}{3\sqrt{10}}\langle 7,4,4,-3\rangle, \frac{1}{3\sqrt{2}}\langle -1,2,2,3\rangle\right\}$

22. for W : $\left\{\frac{1}{\sqrt{14}}\langle 1,-1,1,-1\rangle, \frac{1}{\sqrt{2198}}\langle 19,-5,-9,9\rangle, \frac{1}{\sqrt{125286}}\langle 33,264,-90,-67\rangle\right\}$;
for $W^\perp$: $\left\{\frac{1}{2\sqrt{399}}\langle 3,24,16,6\rangle\right\}$

23. for W : $\left\{\frac{1}{\sqrt{17}}x^2\right\}$; for $W^\perp$: $\left\{\frac{1}{6\sqrt{17}}(7x^2+17x), \frac{1}{2}(x^2+x-2)\right\}$

24. for W : $\left\{\frac{1}{\sqrt{30}}(x^2+1), \frac{1}{\sqrt{330}}(8x^2+15x-7)\right\}$; for $W^\perp$: $\left\{\frac{1}{\sqrt{99}}(4x^2+2x-9)\right\}$

25. for W : $\left\{\sqrt{5}\,x^2, \sqrt{3}\,(5x^2-4x)\right\}$; for $W^\perp$: $\{10x^2-12x+3\}$

26. Start with $\{\langle 5,-2,0\rangle, \vec{e}_1, \vec{e}_3\}$; for V : $\left\{\frac{1}{\sqrt{29}}\langle 5,-2,0\rangle, \frac{1}{\sqrt{29}}\langle 2,5,0\rangle, \vec{e}_3\right\}$;
for W : $\left\{\frac{1}{\sqrt{29}}\langle 5,-2,0\rangle\right\}$; for $W^\perp$: $\left\{\frac{1}{\sqrt{29}}\langle 2,5,0\rangle, \vec{e}_3\right\}$

27. Start with $\{\langle 5,-2,0\rangle, \vec{e}_1, \vec{e}_3\}$; for V : $\left\{\langle 5,-2,0\rangle/\sqrt{120}, \langle 1,2,0\rangle/\sqrt{24}, \langle 0,0,1\rangle/\sqrt{2}\right\}$;
for W : $\left\{\langle 5,-2,0\rangle/\sqrt{120}\right\}$; for $W^\perp$: $\left\{\langle 1,2,0\rangle/\sqrt{24}, \langle 0,0,1\rangle/\sqrt{2}\right\}$

28. Start with $\{\langle 1,-1,0,1\rangle, \langle 1,0,-3,1\rangle, \vec{e}_1, \vec{e}_2\}$;
For V : $\left\{\langle 1,-1,0,1\rangle/\sqrt{3}, \langle 1,2,-9,1\rangle/\sqrt{87}, \langle 19,9,3,-10\rangle/\sqrt{551}, \langle 0,3,1,3\rangle/\sqrt{19}\right\}$
for W : $\left\{\langle 1,-1,0,1\rangle/\sqrt{3}, \langle 1,2,-9,1\rangle/\sqrt{87}\right\}$; for
$W^\perp$: $\left\{\langle 19,9,3,-10\rangle/\sqrt{551}, \langle 0,3,1,3\rangle/\sqrt{19}\right\}$

29. Start with $\{\langle 1,-1,0,1\rangle, \langle 1,0,-3,1\rangle, \vec{e}_1, \vec{e}_2\}$;
For V : $\left\{\frac{1}{\sqrt{11}}\langle 1,-1,0,1\rangle, \frac{1}{\sqrt{3377}}\langle 1,10,-33,1\rangle, \frac{1}{2\sqrt{59865}}\langle 195,108,12,-112\rangle,\right.$
$\left.\frac{1}{\sqrt{104\,409\,309\,290}}\langle -97\,825, 115\,898, 57\,222, 84\,533\rangle\right\}$
for W : $\left\{\frac{1}{\sqrt{11}}\langle 1,-1,0,1\rangle, \frac{1}{\sqrt{3377}}\langle 1,10,-33,1\rangle\right\}$; for $W^\perp$:
$\left\{\frac{1}{2\sqrt{59865}}\langle 195,108,12,-112\rangle, \frac{1}{\sqrt{104\,409\,309\,290}}\langle -97\,825, 115\,898, 57\,222, 84\,533\rangle\right\}$

30. Start with $\{x^2+5x, x^2, 1\}$; for V : $\left\{\frac{1}{\sqrt{72}}(x^2+5x), \frac{1}{\sqrt{8}}(x^2+x), \frac{1}{2}(x^2+x-2)\right\}$;

for W : $\left\{\frac{1}{\sqrt{72}}(x^2+5x)\right\}$; for $W^{\perp}$: $\left\{\frac{1}{\sqrt{8}}(x^2+x), \frac{1}{2}(x^2+x-2)\right\}$

31. Start with $\{x^2-3x, x, 1\}$; for V :
$\left\{\sqrt{\frac{10}{17}}(x^2-3x), \sqrt{\frac{6}{17}}(15x^2-11x), 10x^2-12x+3\right\}$;
for W : $\left\{\sqrt{\frac{10}{17}}(x^2-3x), \sqrt{\frac{6}{17}}(15x^2-11x)\right\}$; for $W^{\perp}$: $\{10x^2-12x+3\}$

33. $\vec{w}_1 = \langle 2, -5/2, 5/2\rangle$, and $\vec{w}_2 = \langle 0, -3/2, -3/2\rangle$ 34. $\vec{w}_1 = \langle 5/2, 0, 5/2\rangle$, and $\vec{w}_2 = \langle -11/2, 5, 11/2\rangle$

35. $\vec{w}_1 = -\frac{5}{4}\langle 1, 1, -1\rangle$, and $\vec{w}_2 = \frac{1}{4}\langle 13, -11, -1\rangle$ 36. $\vec{w}_1 = \frac{1}{22}\langle -71, 118, 165\rangle$, and $\vec{w}_2 = -\frac{1}{22}\langle -5, 8, -11\rangle$

37. $\vec{w}_1 = \left\langle \frac{1}{11}, \frac{2}{11}, -\frac{5}{11}, \frac{5}{11}\right\rangle$, and $\vec{w}_2 = \left\langle \frac{32}{11}, \frac{64}{11}, -\frac{17}{11}, -\frac{49}{11}\right\rangle$

38. $\vec{w}_1 = \left\langle \frac{4}{3}, -\frac{4}{3}, 0, \frac{4}{3}\right\rangle$, and $\vec{w}_2 = \left\langle \frac{11}{3}, -\frac{2}{3}, 7, -\frac{13}{3}\right\rangle$

39. $\vec{w}_1 = \left\langle -\frac{15}{133}, -\frac{120}{133}, -\frac{80}{133}, -\frac{30}{133}\right\rangle$, and $\vec{w}_2 = \left\langle \frac{414}{133}, \frac{918}{133}, -\frac{186}{133}, -\frac{502}{133}\right\rangle$

40. $\vec{w}_1 = -56x^2/17$, and $\vec{w}_2 = 39x^2/17 + 2x - 5$ 41. $\vec{w}_1 = \frac{49}{3}x^2 - 22x$, and $\vec{w}_2 = -\frac{40}{3}x^2 + 16x - 4$

7.5 Exercises

1. a. $\langle \vec{u} \mid \vec{v}\rangle = -100$; b. $\|\vec{u}\| = 3\sqrt{11}$; c. $\|\vec{v}\| = \sqrt{353}$; d. $d(\vec{u}, \vec{v}) = 2\sqrt{163}$
e. $\cos(\theta) = -\frac{100}{3\sqrt{11}\sqrt{353}}$

2. a. $\langle \vec{u} \mid \vec{v}\rangle = -123$; b. $\|\vec{u}\| = \sqrt{38}$; c. $\|\vec{v}\| = \sqrt{429}$; d. $d(\vec{u}, \vec{v}) = \sqrt{713}$
e. $\cos(\theta) = -\frac{123}{\sqrt{38}\sqrt{429}}$

3. a. $\langle \vec{u} \mid \vec{v}\rangle = 78$; b. $\|\vec{u}\| = 6\sqrt{5}$; c. $\|\vec{v}\| = \sqrt{305}$; d. $d(\vec{u}, \vec{v}) = \sqrt{329}$
e. $\cos(\theta) = \frac{13}{5\sqrt{61}}$

4. a. $\langle \vec{u} \mid \vec{v}\rangle = -212$; b. $\|\vec{u}\| = \sqrt{210}$; c. $\|\vec{v}\| = \sqrt{465}$; d. $d(\vec{u}, \vec{v}) = \sqrt{1099}$;
e. $\cos(\theta) = -\frac{212}{\sqrt{97650}}$

5. a. $\langle \vec{u} \mid \vec{v}\rangle = \frac{386}{15}$; b. $\|\vec{u}\| = \frac{1}{15}\sqrt{4245}$; c. $\|\vec{v}\| = \frac{2}{5}\sqrt{230}$; d. $d(\vec{u}, \vec{v}) = \frac{1}{5}\sqrt{105}$;
e. $\cos(\theta) = \frac{193}{\sqrt{39054}}$

6. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ 7. $\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$ 8. $\begin{bmatrix} \frac{9}{11} & -\frac{4}{11} & -\frac{1}{11} & \frac{1}{11} \\ -\frac{4}{11} & \frac{3}{11} & -\frac{2}{11} & \frac{2}{11} \\ -\frac{1}{11} & -\frac{2}{11} & \frac{5}{11} & -\frac{5}{11} \\ \frac{1}{11} & \frac{2}{11} & -\frac{5}{11} & \frac{5}{11} \end{bmatrix}$

9. $\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$ 10. $\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix}$

11. One way is to apply Gram-Schmidt to $\{\langle 3,5,0\rangle, \langle 7,0,5\rangle\}$; we get

$$\begin{bmatrix} \frac{58}{83} & \frac{15}{83} & \frac{35}{83} \\ \frac{15}{83} & \frac{74}{83} & -\frac{21}{83} \\ \frac{35}{83} & -\frac{21}{83} & \frac{34}{83} \end{bmatrix}$$

12. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{16}{65} & \frac{28}{65} \\ 0 & \frac{28}{65} & \frac{49}{65} \end{bmatrix}$

21. c. $f(x) = -7x^4 + 5x^2 - 1$, and $g(x) = 8x^5 - 2x^3 + 6x$

7.6 Exercises

1. $\begin{bmatrix} -8/17 & 15/17 \\ 15/17 & 8/17 \end{bmatrix}$ is improper, while $\begin{bmatrix} -8/17 & -15/17 \\ 15/17 & -8/17 \end{bmatrix}$ is proper.

2. $\begin{bmatrix} 20/29 & -21/29 \\ -21/29 & -20/29 \end{bmatrix}$ is improper, while $\begin{bmatrix} 20/29 & -21/29 \\ 21/29 & 20/29 \end{bmatrix}$ is proper.

3. $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ (improper). 4. $\begin{bmatrix} \frac{1}{2} & \frac{5}{2\sqrt{11}} & \frac{7}{\sqrt{330}} & \frac{1}{\sqrt{30}} \\ -\frac{1}{2} & -\frac{1}{2\sqrt{11}} & \frac{14}{\sqrt{330}} & \frac{2}{\sqrt{30}} \\ \frac{1}{2} & -\frac{3}{2\sqrt{11}} & \frac{-2}{\sqrt{330}} & \frac{4}{\sqrt{30}} \\ -\frac{1}{2} & \frac{3}{2\sqrt{11}} & \frac{-9}{\sqrt{330}} & \frac{3}{\sqrt{30}} \end{bmatrix}$,
(proper)

5. b. $Q = \begin{bmatrix} -20/29 & 21/29 \\ 21/29 & 20/29 \end{bmatrix}$ and $Q' = \begin{bmatrix} 15/17 & -8/17 \\ 8/17 & 15/17 \end{bmatrix}$ c. Q is improper and Q' is proper.

d. $QQ' = \begin{bmatrix} -\frac{132}{493} & \frac{475}{493} \\ \frac{475}{493} & \frac{132}{493} \end{bmatrix}$. e. QQ' is improper. f. $C_{B,B'} = \begin{bmatrix} -\frac{132}{493} & \frac{475}{493} \\ \frac{475}{493} & \frac{132}{493} \end{bmatrix}$

g. $C_{B,B'}$ is improper.

6. There are 2^n possible combinations. 7. There are $n!$ such rearrangements.

15. g. $\left\{ \frac{1}{\sqrt{a^2+b^2}}\langle -b, a, 0\rangle, \frac{1}{\sqrt{a^2+b^2}}\langle -ac, -bc, a^2+b^2\rangle \right\}$

i) $\left\{ \frac{1}{\sqrt{13}}\langle 2,3,0\rangle, \frac{1}{7\sqrt{13}}\langle -18,12,13\rangle, \frac{1}{7}\langle 3,-2,6\rangle \right\}$

7.7 Exercises

1. Q is proper.
Note: other answers are possible in the following if the eigenspace has dimension 2 or bigger.

2. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}; \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$

3. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}; \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$

4. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$

5. $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}; \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$

6. $\begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 10 \end{bmatrix}; \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

7. $\begin{bmatrix} 2-\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2+\sqrt{2} \end{bmatrix}; \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 1/2 & -1/\sqrt{2} & 1/2 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}; \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

9. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}; \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

10. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$; $\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$

11. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$; $\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

12. $\begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -4 \end{bmatrix}$; $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

13. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$; $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$

14. $\begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$; $\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} & 1/2 \\ 0 & 0 & 3/\sqrt{12} & 1/2 \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} & 1/2 \\ 0 & 2/\sqrt{6} & -1/\sqrt{12} & 1/2 \end{bmatrix}$

15. $\begin{bmatrix} -7 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$; $\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} & 1/2 \\ 0 & 0 & 3/\sqrt{12} & 1/2 \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} & 1/2 \\ 0 & 2/\sqrt{6} & -1/\sqrt{12} & 1/2 \end{bmatrix}$

16. $\begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$; $\begin{bmatrix} 0 & -1/2 & -1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 1/2 & 0 & 1/2 \\ 0 & -1/2 & 1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 1/2 & 0 & 1/2 \end{bmatrix}$

17. $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$; $\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \end{bmatrix}$

18. $\begin{bmatrix} -7 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$; $\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \end{bmatrix}$

19. $\begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$; $\begin{bmatrix} -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}$

20. $\begin{bmatrix} -5 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$; $\begin{bmatrix} -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & 0 \end{bmatrix}$

21. $\begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$; $\begin{bmatrix} 0 & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}$

22. $\begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$; $\begin{bmatrix} 0 & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}$

There is exactly one different eigenvalue between the matrices in Exercise 21 and 22, but the diagonalizing orthogonal matrix is the same for both. This is explained further in Exercises 26 and 27.

23. a) Eigenvalues $a + 2b$, with multiplicity 1, and $a - b$, with multiplicity 2.
b) Eigenvalues $a + 3b$, with multiplicity 1, and $a - b$, with multiplicity 3.

24. a) Eigenvalues $\pm c_1$ and $\pm c_2$. b) Eigenvalues $\pm c_i$, all with multiplicity 1.

25. b) Eigenvalues $\pm c_1, \pm c_2, \ldots \pm c_{k-1}, c_k$, all with multiplicity 1, where $k = (n+1)/2$.

26. b) Eigenvalues $a \pm b$, each with multiplicity $n/2$.

27. c) Eigenvalues a, with multiplicity 1, and $a \pm b$, each with multiplicity $(n-1)/2$.
d) Eigenvalues b, with multiplicity 1, and $a \pm b$, each with multiplicity $(n-1)/2$.

30. $Q = \begin{bmatrix} -\frac{1}{5}\sqrt{5} & \frac{1}{15}\sqrt{30} & -\frac{1}{3}\sqrt{6} \\ \frac{2}{5}\sqrt{5} & \frac{1}{30}\sqrt{30} & -\frac{1}{6}\sqrt{6} \\ 0 & \frac{1}{6}\sqrt{30} & \frac{1}{6}\sqrt{6} \end{bmatrix}$; $D = \begin{bmatrix} -12 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & 18 \end{bmatrix}$

31. $Q = \begin{bmatrix} -\frac{1}{5}\sqrt{5} & \frac{1}{15}\sqrt{30} & -\frac{1}{3}\sqrt{6} \\ \frac{2}{5}\sqrt{5} & \frac{1}{30}\sqrt{30} & -\frac{1}{6}\sqrt{6} \\ 0 & \frac{1}{6}\sqrt{30} & \frac{1}{6}\sqrt{6} \end{bmatrix}$; $D = \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

32. $Q = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{3} & \frac{1}{6}\sqrt{6} \\ \frac{1}{2}\sqrt{2} & \frac{1}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} \\ 0 & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{6} \end{bmatrix}$; $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

33. $Q = \begin{bmatrix} -\frac{1}{14}\sqrt{14} & \frac{3}{10}\sqrt{10} & \frac{1}{35}\sqrt{35} \\ \frac{3}{14}\sqrt{14} & \frac{1}{10}\sqrt{10} & -\frac{3}{35}\sqrt{35} \\ \frac{1}{7}\sqrt{14} & 0 & \frac{1}{7}\sqrt{35} \end{bmatrix}$; $D = \begin{bmatrix} -35 & 0 & 0 \\ 0 & -21 & 0 \\ 0 & 0 & -21 \end{bmatrix}$

34. $Q = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{3} & \frac{1}{6}\sqrt{6} \\ \frac{1}{2}\sqrt{2} & \frac{1}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} \\ 0 & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{6} \end{bmatrix}$; $D = \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

35. $Q = \begin{bmatrix} -\frac{1}{14}\sqrt{14} & \frac{3}{10}\sqrt{10} & \frac{1}{35}\sqrt{35} \\ \frac{3}{14}\sqrt{14} & \frac{1}{10}\sqrt{10} & -\frac{3}{35}\sqrt{35} \\ \frac{1}{7}\sqrt{14} & 0 & \frac{1}{7}\sqrt{35} \end{bmatrix}$; $D = \begin{bmatrix} -98 & 0 & 0 \\ 0 & 56 & 0 \\ 0 & 0 & 56 \end{bmatrix}$

36. $Q = \begin{bmatrix} -\frac{1}{14}\sqrt{14} & \frac{3}{10}\sqrt{10} & \frac{1}{35}\sqrt{35} \\ \frac{3}{14}\sqrt{14} & \frac{1}{10}\sqrt{10} & -\frac{3}{35}\sqrt{35} \\ \frac{1}{7}\sqrt{14} & 0 & \frac{1}{7}\sqrt{35} \end{bmatrix}$; $D = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

37. $Q = \begin{bmatrix} -\frac{1}{14}\sqrt{14} & \frac{3}{10}\sqrt{10} & \frac{1}{35}\sqrt{35} \\ \frac{3}{14}\sqrt{14} & \frac{1}{10}\sqrt{10} & -\frac{3}{35}\sqrt{35} \\ \frac{1}{7}\sqrt{14} & 0 & \frac{1}{7}\sqrt{35} \end{bmatrix}$; $D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

38. $Q = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{3} & \frac{1}{6}\sqrt{6} \\ \frac{1}{2}\sqrt{2} & \frac{1}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} \\ 0 & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{6} \end{bmatrix}$; $D = \begin{bmatrix} -12 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & 30 \end{bmatrix}$

39. $Q = \begin{bmatrix} -\frac{3}{83}\sqrt{83} & -\frac{12}{2905}\sqrt{8715} & \frac{1}{11}\sqrt{22} & \frac{6}{385}\sqrt{2310} \\ \frac{5}{83}\sqrt{83} & -\frac{23}{8715}\sqrt{8715} & -\frac{3}{22}\sqrt{22} & \frac{23}{2310}\sqrt{2310} \\ \frac{7}{83}\sqrt{83} & \frac{1}{8715}\sqrt{8715} & \frac{3}{22}\sqrt{22} & -\frac{1}{2310}\sqrt{2310} \\ 0 & \frac{1}{105}\sqrt{8715} & 0 & \frac{1}{105}\sqrt{2310} \end{bmatrix};$

$D = \begin{bmatrix} -42 & 0 & 0 & 0 \\ 0 & -42 & 0 & 0 \\ 0 & 0 & 63 & 0 \\ 0 & 0 & 0 & 63 \end{bmatrix}$

40. $Q = \begin{bmatrix} \frac{3}{11}\sqrt{11} & -\frac{1}{77}\sqrt{231} & \frac{1}{10}\sqrt{10} & \frac{1}{70}\sqrt{210} \\ -\frac{1}{11}\sqrt{11} & \frac{1}{231}\sqrt{231} & \frac{3}{10}\sqrt{10} & -\frac{1}{210}\sqrt{210} \\ \frac{1}{11}\sqrt{11} & \frac{10}{231}\sqrt{231} & 0 & -\frac{1}{21}\sqrt{210} \\ 0 & \frac{1}{21}\sqrt{231} & 0 & \frac{1}{21}\sqrt{210} \end{bmatrix}; D = \begin{bmatrix} -63 & 0 & 0 & 0 \\ 0 & -63 & 0 & 0 \\ 0 & 0 & 21 & 0 \\ 0 & 0 & 0 & 21 \end{bmatrix}$

41. $Q = \begin{bmatrix} \frac{3}{11}\sqrt{11} & -\frac{1}{77}\sqrt{231} & \frac{1}{10}\sqrt{10} & \frac{1}{70}\sqrt{210} \\ -\frac{1}{11}\sqrt{11} & \frac{1}{231}\sqrt{231} & \frac{3}{10}\sqrt{10} & -\frac{1}{210}\sqrt{210} \\ \frac{1}{11}\sqrt{11} & \frac{10}{231}\sqrt{231} & 0 & -\frac{1}{21}\sqrt{210} \\ 0 & \frac{1}{21}\sqrt{231} & 0 & \frac{1}{21}\sqrt{210} \end{bmatrix}; D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 21 & 0 \\ 0 & 0 & 0 & 21 \end{bmatrix}$

42. $Q = \begin{bmatrix} \frac{13}{203}\sqrt{203} & -\frac{1}{203}\sqrt{1015} & \frac{1}{42}\sqrt{42} & \frac{1}{42}\sqrt{210} \\ -\frac{5}{203}\sqrt{203} & -\frac{1}{145}\sqrt{1015} & \frac{5}{42}\sqrt{42} & \frac{1}{30}\sqrt{210} \\ \frac{3}{203}\sqrt{203} & \frac{2}{203}\sqrt{1015} & \frac{2}{21}\sqrt{42} & -\frac{1}{21}\sqrt{210} \\ 0 & \frac{1}{35}\sqrt{1015} & 0 & \frac{1}{35}\sqrt{210} \end{bmatrix}; D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 35 & 0 \\ 0 & 0 & 0 & 35 \end{bmatrix}$

43. $Q = \begin{bmatrix} \frac{3}{58}\sqrt{58} & -\frac{7}{899}\sqrt{899} & -\frac{1}{186}\sqrt{434} & \frac{1}{3}\sqrt{7} \\ \frac{7}{58}\sqrt{58} & \frac{3}{899}\sqrt{899} & \frac{1}{434}\sqrt{434} & -\frac{1}{7}\sqrt{7} \\ 0 & \frac{1}{31}\sqrt{899} & -\frac{1}{651}\sqrt{434} & \frac{2}{21}\sqrt{7} \\ 0 & 0 & \frac{1}{21}\sqrt{434} & \frac{1}{21}\sqrt{7} \end{bmatrix}; D = \begin{bmatrix} -63 & 0 & 0 & 0 \\ 0 & -63 & 0 & 0 \\ 0 & 0 & -63 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

44. $Q = \begin{bmatrix} -\frac{1}{7}\sqrt{14} & \frac{8}{287}\sqrt{574} & -\frac{1}{1148}\sqrt{574} & -\frac{2}{15}\sqrt{15} & \frac{1}{420}\sqrt{210} \\ -\frac{3}{14}\sqrt{14} & -\frac{11}{574}\sqrt{574} & -\frac{1}{164}\sqrt{574} & \frac{1}{15}\sqrt{15} & \frac{1}{60}\sqrt{210} \\ \frac{1}{14}\sqrt{14} & -\frac{1}{574}\sqrt{574} & -\frac{23}{1148}\sqrt{574} & -\frac{1}{15}\sqrt{15} & \frac{23}{420}\sqrt{210} \\ 0 & \frac{1}{41}\sqrt{574} & -\frac{3}{574}\sqrt{574} & \frac{1}{5}\sqrt{15} & \frac{1}{70}\sqrt{210} \\ 0 & 0 & \frac{1}{28}\sqrt{574} & 0 & \frac{1}{28}\sqrt{210} \end{bmatrix};$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 56 & 0 \\ 0 & 0 & 0 & 0 & 56 \end{bmatrix}$$

7.8 Exercises

1. The rref of $\begin{bmatrix} 3 & -1 & -6 & 2 \\ -2 & 1 & 3 & -2 \\ 5 & -2 & -9 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, so the system is inconsistent.

$C = \begin{bmatrix} 3 & -1 \\ -2 & 1 \\ 5 & -2 \end{bmatrix}$, $C^{\top}C = \begin{bmatrix} 38 & -15 \\ -15 & 6 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\vec{x}_1 = \langle 0, -1, 0 \rangle$,

the nullspace has basis $\{\langle 3, 3, 1, 0 \rangle\}$,

$A\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \vec{b}_1$, $[proj_W] = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$, $\vec{b} - \vec{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$,

the common error is $\sqrt{3}$.

2. The rref of $\begin{bmatrix} 1 & 1 & -1 & -2 \\ 1 & -2 & 5 & 9 \\ 2 & -1 & 4 & 5 \\ 2 & 1 & 0 & 4 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the system is inconsistent.

$C = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 2 & -1 \\ 2 & 1 \end{bmatrix}$, $C^{\top}C = \begin{bmatrix} 10 & -1 \\ -1 & 7 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} \frac{154}{69} \\ -\frac{185}{69} \end{bmatrix}$, $\vec{x}_1 = \left\langle \frac{154}{69}, -\frac{185}{69}, 0 \right\rangle$,

the nullspace has basis $\{\langle -1, 2, 1 \rangle\}$,

$A\vec{x}_1 = \begin{bmatrix} -\frac{31}{69} \\ \frac{524}{69} \\ \frac{493}{69} \\ \frac{41}{23} \end{bmatrix} = \vec{b}_1$, $[proj_W] = \frac{1}{69}\begin{bmatrix} 19 & -14 & 5 & 27 \\ -14 & 43 & 29 & -9 \\ 5 & 29 & 34 & 18 \\ 27 & -9 & 18 & 42 \end{bmatrix}$, $\vec{b} - \vec{b}_1 = \begin{bmatrix} -\frac{107}{69} \\ \frac{97}{69} \\ -\frac{148}{69} \\ \frac{51}{23} \end{bmatrix}$,

the common error is $\frac{1}{69}\sqrt{66171} \approx 3.7281$

3. The rref of $\begin{bmatrix} 3 & -15 & -6 & 2 & 28 \\ -2 & 10 & 4 & -4 & -26 \\ 5 & -25 & -10 & -1 & 13 \end{bmatrix}$ is $\begin{bmatrix} 1 & -5 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$,

$C = \begin{bmatrix} 3 & 2 \\ -2 & -4 \\ 5 & -1 \end{bmatrix}$, $C^{\top}C = \begin{bmatrix} 38 & 9 \\ 9 & 21 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} \frac{966}{239} \\ \frac{1259}{239} \end{bmatrix}$, $\vec{x}_1 = \left\langle \frac{966}{239}, 0, 0, \frac{1259}{239} \right\rangle$,

the nullspace has basis $\{\langle 5, 1, 0, 0\rangle, \langle 2, 0, 1, 0\rangle\}$,

$A\vec{x}_1 = \begin{bmatrix} \frac{5416}{239} \\ -\frac{6968}{239} \\ \frac{3571}{239} \end{bmatrix} = \vec{b}_1$, $[proj_W] = \frac{1}{717}\begin{bmatrix} 233 & -286 & 176 \\ -286 & 548 & 104 \\ 176 & 104 & 653 \end{bmatrix}$,

$\vec{b} - \vec{b}_1 = \begin{bmatrix} \frac{1276}{239} \\ \frac{754}{239} \\ -\frac{464}{239} \end{bmatrix}$,

the common error is $\frac{58}{239}\sqrt{717} \approx 6.4981$.

4. The rref of $\begin{bmatrix} 3 & -2 & 19 & 4 & 38 \\ 4 & -1 & 22 & -3 & 5 \\ -1 & 5 & -15 & 2 & -28 \\ 1 & 2 & 1 & 4 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 5 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, so the system is inconsistent.

$C = \begin{bmatrix} 3 & -2 & 4 \\ 4 & -1 & -3 \\ -1 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, $C^{\top}C = \begin{bmatrix} 27 & -13 & 2 \\ -13 & 34 & 13 \\ 2 & 13 & 45 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} \frac{34762}{14165} \\ -\frac{98068}{14165} \\ \frac{54801}{14165} \end{bmatrix}$,

$\vec{x}_1 = \left\langle \frac{34762}{14165}, -\frac{98068}{14165}, 0, \frac{54801}{14165} \right\rangle$, the nullspace has basis $\{\langle -5, 2, 1, 0\rangle\}$,

$A\vec{x}_1 = \begin{bmatrix} \frac{519626}{14165} \\ \frac{72713}{14165} \\ -\frac{83100}{2833} \\ \frac{11566}{2833} \end{bmatrix} = \vec{b}_1$, $[proj_W] = \frac{1}{28330}\begin{bmatrix} 22089 & 632 & -6320 & 9875 \\ 632 & 28266 & 640 & -1000 \\ -6320 & 640 & 21930 & 10000 \\ 9875 & -1000 & 10000 & 12705 \end{bmatrix}$,

$\vec{b} - \vec{b}_1 = \begin{bmatrix} \frac{18644}{14165} \\ -\frac{1888}{14165} \\ \frac{3776}{2833} \\ -\frac{5900}{2833} \end{bmatrix}$, the common error is $\frac{236}{14165}\sqrt{28330} \approx 2.8043$.

5. The rref of $\begin{bmatrix} 4 & -3 & 1 & 11 \\ -2 & 0 & -5 & -9 \\ 3 & 1 & 2 & 5 \\ -1 & 5 & 6 & -7 \\ 0 & 3 & 2 & -4 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the system is inconsistent.

$C = \begin{bmatrix} 4 & -3 & 1 \\ -2 & 0 & -5 \\ 3 & 1 & 2 \\ -1 & 5 & 6 \\ 0 & 3 & 2 \end{bmatrix}$, $C^{\top}C = \begin{bmatrix} 30 & -14 & 14 \\ -14 & 44 & 35 \\ 14 & 35 & 70 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} \frac{1991}{1399} \\ -\frac{2804}{1399} \\ \frac{9265}{9793} \end{bmatrix} = \vec{x}_1$,

the nullspace is $\left\{\vec{0}_4\right\}$, so $\vec{x}_1$ is a unique solution,

$A\vec{x}_1 = \begin{bmatrix} \frac{123897}{9793} \\ -\frac{74199}{9793} \\ \frac{40713}{9793} \\ -\frac{56487}{9793} \\ -\frac{40354}{9793} \end{bmatrix} = \vec{b}_1$,

$[proj_W] = \frac{1}{9793}\begin{bmatrix} 11288 & -5998 & 5510 & -4666 & -2358 \\ -5998 & 13400 & 1934 & -5818 & 3054 \\ 5510 & 1934 & 13659 & 391 & 6834 \\ -4666 & -5818 & 391 & 13731 & 4962 \\ -2358 & 3054 & 6834 & 4962 & 6680 \end{bmatrix}$,

$\vec{b} - \vec{b}_1 = \begin{bmatrix} -\frac{16174}{9793} \\ -\frac{13938}{9793} \\ \frac{8252}{9793} \\ -\frac{12064}{9793} \\ \frac{1182}{9793} \end{bmatrix}$, the common error is $\frac{6}{9793}\sqrt{18636079} \approx 2.6449$.

9. $C = \begin{bmatrix} 7 & -4 \\ 3 & 0 \\ 0 & 3 \end{bmatrix}$ (other choices are possible), so $C^{\top}C = \begin{bmatrix} 58 & -28 \\ -28 & 25 \end{bmatrix}$, and

$$[proj_{\Pi}] = \begin{bmatrix} 7 & -4 \\ 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 58 & -28 \\ -28 & 25 \end{bmatrix}^{-1} \begin{bmatrix} 7 & 3 & 0 \\ -4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{65}{74} & \frac{21}{74} & -\frac{6}{37} \\ \frac{21}{74} & \frac{25}{74} & \frac{14}{37} \\ -\frac{6}{37} & \frac{14}{37} & \frac{29}{37} \end{bmatrix}$$

7.9 Exercises

1. $\begin{bmatrix} \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \\ \frac{2}{5}\sqrt{5} & -\frac{1}{5}\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \frac{13}{5}\sqrt{5} \\ 0 & \frac{1}{5}\sqrt{5} \end{bmatrix}$ 2. $\begin{bmatrix} \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} \\ \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$

3. $\begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{6}\sqrt{2} \\ 0 & \frac{2}{3}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{6}\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\frac{1}{2}\sqrt{2} \\ 0 & \frac{3}{2}\sqrt{2} \end{bmatrix}$ 4.

$$\begin{bmatrix} \frac{1}{11}\sqrt{11} & -\frac{9}{682}\sqrt{682} \\ 0 & \frac{1}{31}\sqrt{682} \\ -\frac{1}{11}\sqrt{11} & \frac{9}{682}\sqrt{682} \\ \frac{3}{11}\sqrt{11} & \frac{3}{341}\sqrt{682} \end{bmatrix} \begin{bmatrix} \sqrt{11} & -\frac{2}{11}\sqrt{11} \\ 0 & \frac{1}{11}\sqrt{682} \end{bmatrix}$$

5. $\begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{6}\sqrt{2} & -\frac{2}{3} \\ 0 & \frac{2}{3}\sqrt{2} & -\frac{1}{3} \\ \frac{1}{2}\sqrt{2} & \frac{1}{6}\sqrt{2} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\frac{1}{2}\sqrt{2} & \frac{3}{2}\sqrt{2} \\ 0 & \frac{3}{2}\sqrt{2} & \frac{7}{6}\sqrt{2} \\ 0 & 0 & \frac{5}{3} \end{bmatrix}$

6. $\begin{bmatrix} \frac{\sqrt{11}}{11} & -\frac{9\sqrt{682}}{682} & \frac{43\sqrt{2418}}{2418} \\ 0 & \frac{\sqrt{682}}{31} & \frac{2\sqrt{2418}}{403} \\ -\frac{\sqrt{11}}{11} & \frac{9\sqrt{682}}{682} & \frac{19\sqrt{2418}}{2418} \\ \frac{3\sqrt{11}}{11} & \frac{3\sqrt{682}}{341} & -\frac{4\sqrt{2418}}{1209} \end{bmatrix} \begin{bmatrix} \sqrt{11} & -\frac{2\sqrt{11}}{11} & -\frac{4\sqrt{11}}{11} \\ 0 & \frac{\sqrt{682}}{11} & \frac{25\sqrt{682}}{682} \\ 0 & 0 & \frac{\sqrt{2418}}{62} \end{bmatrix}$

7. $\begin{bmatrix} \frac{\sqrt{11}}{11} & -\frac{9\sqrt{682}}{682} & \frac{43\sqrt{2418}}{2418} & \frac{\sqrt{39}}{39} \\ 0 & \frac{\sqrt{682}}{31} & \frac{2\sqrt{2418}}{403} & \frac{\sqrt{39}}{13} \\ -\frac{\sqrt{11}}{11} & \frac{9\sqrt{682}}{682} & \frac{19\sqrt{2418}}{2418} & -\frac{5\sqrt{39}}{39} \\ \frac{3\sqrt{11}}{11} & \frac{3\sqrt{682}}{341} & -\frac{4\sqrt{2418}}{1209} & -\frac{2\sqrt{39}}{39} \end{bmatrix} \begin{bmatrix} \sqrt{11} & -\frac{2\sqrt{11}}{11} & -\frac{4\sqrt{11}}{11} & \frac{8\sqrt{11}}{11} \\ 0 & \frac{\sqrt{682}}{11} & \frac{25\sqrt{682}}{682} & \frac{8\sqrt{682}}{341} \\ 0 & 0 & \frac{\sqrt{2418}}{62} & \frac{10\sqrt{2418}}{1209} \\ 0 & 0 & 0 & \frac{5\sqrt{39}}{39} \end{bmatrix}$

8. $$\begin{bmatrix} \frac{\sqrt{7}}{7} & -\frac{\sqrt{231}}{77} & \frac{4\sqrt{33}}{33} \\ 0 & \frac{2\sqrt{231}}{33} & \frac{\sqrt{33}}{22} \\ -\frac{\sqrt{7}}{7} & \frac{\sqrt{231}}{77} & \frac{\sqrt{33}}{22} \\ \frac{\sqrt{7}}{7} & \frac{4\sqrt{231}}{231} & -\frac{7\sqrt{33}}{66} \\ \frac{2\sqrt{7}}{7} & \frac{\sqrt{231}}{231} & \frac{\sqrt{33}}{66} \end{bmatrix} \begin{bmatrix} \sqrt{7} & -\frac{4\sqrt{7}}{7} & -\frac{4\sqrt{7}}{7} \\ 0 & \frac{\sqrt{231}}{7} & \frac{4\sqrt{231}}{77} \\ 0 & 0 & \frac{2\sqrt{33}}{11} \end{bmatrix}$$

9. $$\begin{bmatrix} \frac{\sqrt{7}}{7} & -\frac{\sqrt{231}}{77} & \frac{4\sqrt{33}}{33} & \frac{\sqrt{30}}{10} \\ 0 & \frac{2\sqrt{231}}{33} & \frac{\sqrt{33}}{22} & \frac{\sqrt{30}}{60} \\ -\frac{\sqrt{7}}{7} & \frac{\sqrt{231}}{77} & \frac{\sqrt{33}}{22} & -\frac{\sqrt{30}}{20} \\ \frac{\sqrt{7}}{7} & \frac{4\sqrt{231}}{231} & -\frac{7\sqrt{33}}{66} & \frac{\sqrt{30}}{12} \\ \frac{2\sqrt{7}}{7} & \frac{\sqrt{231}}{231} & \frac{\sqrt{33}}{66} & -\frac{7\sqrt{30}}{60} \end{bmatrix} \begin{bmatrix} \sqrt{7} & -\frac{4\sqrt{7}}{7} & -\frac{4\sqrt{7}}{7} & \frac{4\sqrt{7}}{7} \\ 0 & \frac{\sqrt{231}}{7} & \frac{4\sqrt{231}}{77} & \frac{16\sqrt{231}}{231} \\ 0 & 0 & \frac{2\sqrt{33}}{11} & -\frac{\sqrt{33}}{11} \\ 0 & 0 & 0 & \frac{\sqrt{30}}{3} \end{bmatrix}$$

We will show in the answers below only the QR-decomposition of C, the matrix consisting of the linearly independent columns of A, and how to obtain the unique solution $\vec{x} = R^{-1}Q^{\top}\vec{b}$ to the normal system. The rest of the solution is shown in the answers to 7.8:

10. $$C = \begin{bmatrix} \frac{3\sqrt{38}}{38} & \frac{7\sqrt{114}}{114} \\ -\frac{\sqrt{38}}{19} & \frac{4\sqrt{114}}{57} \\ \frac{5\sqrt{38}}{38} & -\frac{\sqrt{114}}{114} \end{bmatrix} \begin{bmatrix} \sqrt{38} & -\frac{15\sqrt{38}}{38} \\ 0 & \frac{\sqrt{114}}{38} \end{bmatrix};$$

$$\begin{bmatrix} \frac{\sqrt{38}}{38} & \frac{5\sqrt{114}}{38} \\ 0 & \frac{\sqrt{114}}{3} \end{bmatrix} \begin{bmatrix} \frac{3\sqrt{38}}{38} & -\frac{\sqrt{38}}{19} & \frac{5\sqrt{38}}{38} \\ \frac{7\sqrt{114}}{114} & \frac{4\sqrt{114}}{57} & -\frac{\sqrt{114}}{114} \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \vec{x}_1$$

11. $$C = \begin{bmatrix} \frac{\sqrt{10}}{10} & \frac{11\sqrt{690}}{690} \\ \frac{\sqrt{10}}{10} & -\frac{19\sqrt{690}}{690} \\ \frac{\sqrt{10}}{5} & -\frac{4\sqrt{690}}{345} \\ \frac{\sqrt{10}}{5} & \frac{2\sqrt{690}}{115} \end{bmatrix} \begin{bmatrix} \sqrt{10} & -\frac{\sqrt{10}}{10} \\ 0 & \frac{\sqrt{690}}{10} \end{bmatrix};$$

$$\begin{bmatrix} \frac{\sqrt{10}}{10} & \frac{\sqrt{690}}{690} \\ 0 & \frac{\sqrt{690}}{69} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{5} & \frac{\sqrt{10}}{5} \\ \frac{11\sqrt{690}}{690} & -\frac{19\sqrt{690}}{690} & -\frac{4\sqrt{690}}{345} & \frac{2\sqrt{690}}{115} \end{bmatrix} \begin{bmatrix} -2 \\ 9 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{154}{69} \\ -\frac{185}{69} \end{bmatrix} = \vec{x}$$

12. $$C = \begin{bmatrix} 3 & -15 & -6 & 2 \\ -2 & 10 & 4 & -4 \\ 5 & -25 & -10 & -1 \end{bmatrix} \begin{bmatrix} \frac{3\sqrt{38}}{38} & \frac{49\sqrt{27246}}{27246} \\ -\frac{\sqrt{38}}{19} & -\frac{67\sqrt{27246}}{13623} \\ \frac{5\sqrt{38}}{38} & -\frac{83\sqrt{27246}}{27246} \end{bmatrix} \begin{bmatrix} \sqrt{38} & \frac{9\sqrt{38}}{38} \\ 0 & \frac{\sqrt{27246}}{38} \end{bmatrix};$$

$$\begin{bmatrix} \frac{\sqrt{38}}{38} & -\frac{3\sqrt{27246}}{9082} \\ 0 & \frac{\sqrt{27246}}{717} \end{bmatrix} \begin{bmatrix} \frac{3\sqrt{38}}{38} & -\frac{\sqrt{38}}{19} & \frac{5\sqrt{38}}{38} \\ \frac{49\sqrt{27246}}{27246} & -\frac{67\sqrt{27246}}{13623} & -\frac{83\sqrt{27246}}{27246} \end{bmatrix} \begin{bmatrix} 28 \\ -26 \\ 13 \end{bmatrix} = \begin{bmatrix} \frac{966}{239} \\ \frac{1259}{239} \end{bmatrix} =$$

13. $$C = \begin{bmatrix} \frac{\sqrt{3}}{3} & -\frac{5\sqrt{2247}}{2247} & \frac{3039\sqrt{21219170}}{21219170} \\ \frac{4\sqrt{3}}{9} & \frac{25\sqrt{2247}}{6741} & -\frac{1409\sqrt{21219170}}{10609585} \\ -\frac{\sqrt{3}}{9} & \frac{122\sqrt{2247}}{6741} & -\frac{15\sqrt{21219170}}{2121917} \\ \frac{\sqrt{3}}{9} & \frac{67\sqrt{2247}}{6741} & \frac{401\sqrt{21219170}}{4243834} \end{bmatrix} \begin{bmatrix} 3\sqrt{3} & -\frac{13\sqrt{3}}{9} & \frac{2\sqrt{3}}{9} \\ 0 & \frac{\sqrt{2247}}{9} & \frac{377\sqrt{2247}}{6741} \\ 0 & 0 & \frac{\sqrt{21219170}}{749} \end{bmatrix};$$

$$\begin{bmatrix} \frac{\sqrt{3}}{9} & \frac{13\sqrt{2247}}{6741} & -\frac{237\sqrt{21219170}}{21219170} \\ 0 & \frac{3\sqrt{2247}}{749} & -\frac{377\sqrt{21219170}}{21219170} \\ 0 & 0 & \frac{\sqrt{21219170}}{28330} \end{bmatrix} \cdot$$

$$\begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{4\sqrt{3}}{9} & -\frac{\sqrt{3}}{9} & \frac{\sqrt{3}}{9} \\ -\frac{5\sqrt{2247}}{2247} & \frac{25\sqrt{2247}}{6741} & \frac{122\sqrt{2247}}{6741} & \frac{67\sqrt{2247}}{6741} \\ \frac{3039\sqrt{21219170}}{21219170} & -\frac{1409\sqrt{21219170}}{10609585} & -\frac{15\sqrt{21219170}}{2121917} & \frac{401\sqrt{21219170}}{4243834} \end{bmatrix} \cdot$$

$$\begin{bmatrix} 38 \\ 5 \\ -28 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{34762}{14165} \\ -\frac{98068}{14165} \\ \frac{54801}{14165} \end{bmatrix} = \vec{x}_1$$

14. $$A = C = \begin{bmatrix} \frac{2\sqrt{30}}{15} & -\frac{17\sqrt{8430}}{8430} & \frac{219\sqrt{5503666}}{5503666} \\ -\frac{\sqrt{30}}{15} & -\frac{7\sqrt{8430}}{4215} & -\frac{852\sqrt{5503666}}{2751833} \\ \frac{\sqrt{30}}{10} & \frac{6\sqrt{8430}}{1405} & -\frac{579\sqrt{5503666}}{2751833} \\ -\frac{\sqrt{30}}{30} & \frac{34\sqrt{8430}}{4215} & \frac{405\sqrt{5503666}}{2751833} \\ 0 & \frac{3\sqrt{8430}}{562} & -\frac{745\sqrt{5503666}}{5503666} \end{bmatrix} \begin{bmatrix} \sqrt{30} & -\frac{7\sqrt{30}}{15} & \frac{7\sqrt{30}}{15} \\ 0 & \frac{\sqrt{8430}}{15} & \frac{623\sqrt{8430}}{8430} \\ 0 & 0 & \frac{\sqrt{5503666}}{562} \end{bmatrix};$$

$$\begin{bmatrix} \frac{\sqrt{30}}{30} & \frac{7\sqrt{8430}}{8430} & -\frac{79\sqrt{5503666}}{786238} \\ 0 & \frac{\sqrt{8430}}{562} & -\frac{89\sqrt{5503666}}{786238} \\ 0 & 0 & \frac{\sqrt{5503666}}{9793} \end{bmatrix} \cdot$$

$$\begin{bmatrix} \frac{2\sqrt{30}}{15} & -\frac{\sqrt{30}}{15} & \frac{\sqrt{30}}{10} & -\frac{\sqrt{30}}{30} & 0 \\ -\frac{17\sqrt{8430}}{8430} & -\frac{7\sqrt{8430}}{4215} & \frac{6\sqrt{8430}}{1405} & \frac{34\sqrt{8430}}{4215} & \frac{3\sqrt{8430}}{562} \\ \frac{219\sqrt{5503666}}{5503666} & -\frac{852\sqrt{5503666}}{2751833} & -\frac{579\sqrt{5503666}}{2751833} & \frac{405\sqrt{5503666}}{2751833} & -\frac{745\sqrt{5503666}}{5503666} \end{bmatrix} \cdot$$

$$\begin{bmatrix} 11 \\ -9 \\ 5 \\ -7 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{1991}{1399} \\ -\frac{2804}{1399} \\ \frac{9265}{9793} \end{bmatrix} = \vec{x}_1$$