Errata and Addenda from *A Portrait of Linear Algebra*, 3rd Edition, 2016;
Updated 2018/10/03

Chapter Zero:

p. 14. An integer $b \in \mathbb{Z}$ is ***odd*** if there exists $d \in \mathbb{Z}$ such that $b = 2d + 1$.

Exercise 31 should be: For all $x,\ y \in \mathbb{R}$: if $x \cdot y$ is irrational, then either x is irrational ***or*** y is irrational.

Exercise 55 (b): $N = N_1 \cdot N_2$

Section 1.2 Exercise 26: refers to Exercise 21 and 22 in Section 1.1, not Exercises 18 and 19.

Section 1.3, page 60:

Since this is true for ***any*** r and s, let us first substitute $s = \vec{u} \circ \vec{u}$. Then we get:

Section 1.4. Exercise 9. The matrix is not reduced because of the 7 in the bottom row. This should be changed to a 1. The conclusion is still correct: there are no solutions to this system.

Section 1.5

Exercise 7. Change the third equation to $-2x + y - 4z = 7$.
The new system will be inconsistent and square.
Original system is consistent and square.

Exercise 22. Change S to: $S = \{\langle 5, 3, -6, -2\rangle, \langle -3, -2, 5, 2\rangle, \langle -1, -2, 11, 6\rangle, \langle -5, -4, 1, 6\rangle\}$
The corrected 3rd vector will yield the correct answer in the Key.

Exercise 25. Change S to: $S = \{\langle 5, 3, -6, -2\rangle, \langle -3, -2, 5, 2\rangle, \langle 1, 0, 3, 2\rangle, \langle -4, -1, -5, -4\rangle\}$
The corrected 3rd and 4th vectors will yield the correct answer in the Key.

Section 1.5

Exercise 44. The current notation is somewhat confusing. We can rewrite it as follows:
In this Exercise, we will guide you to prove that if A is an $m \times n$ matrix, $\vec{b}$ is an $m \times 1$ matrix, and we can find ***at least two distinct solutions*** to the system $A\vec{x} = \vec{b}$, then we can find an ***infinite*** number of solutions to this system.

a. First, show that if $\vec{x}_1$ and $\vec{x}_2$ are two such ***distinct*** solutions to $A\vec{x} = \vec{b}$, then $\vec{w} = \vec{x}_1 - \vec{x}_2$ is a solution of the ***homogenous*** system $A\vec{w} = \vec{0}_m$.

b. Next, we will use scalar multiples of $\vec{x}_1 - \vec{x}_2$ to construct an infinite number of solutions to the homogeneous system $A\vec{w} = \vec{0}_m$ in (a). Explain why $t_1(\vec{x}_1 - \vec{x}_2) \neq t_2(\vec{x}_1 - \vec{x}_2)$ if $t_1 \neq t_2$. You may want to use the Zero Factors Theorem. Explain why this means that the set $\{t(\vec{x}_1 - \vec{x}_2) | t \in \mathbb{R}\}$ is an infinite set, and every vector in this set is a solution to $A\vec{w} = \vec{0}_m$.

c. Use $\vec{x}_1$ and part (b) to construct an infinite number of solutions to the original system $A\vec{x} = \vec{b}$, and show that the vectors you constructed are indeed solutions to this system.

d. As a bonus, prove that any linear system $A\vec{x} = \vec{b}$ either has: (1) no solutions, (2) exactly one solution, or (3) an infinite number of solutions. Hint: Use a Case-by-Case Analysis, but be careful how you begin the 3rd Case.

Section 1.6, page 104. $\mathbf{-3}\vec{v}_1 + \mathbf{2}\vec{v}_1$ should read $\mathbf{-3}\vec{v}_1 + \mathbf{2}\vec{v}_2$.

Section 1.6 Exercise 38. Change $\vec{v}_5$ to $\langle 3, -6, 7\rangle$.
This corrected vector will yield the correct answer in the Key.

Section 1.8, page 127. Middle of the page, below highlighted pink line: should read. . . we can also write $\vec{c}_3$, $\vec{c}_4$, and $\vec{c}_6$ in terms of these three columns . . .

Section 1.8, page 127. Middle of the page, below highlighted pink line: should read. . . If we were to arrange these three vectors . . .

Section 1.8 Exercise 68.

My apologies for this major rewrite. To make induction work, we must begin with the ***bottom-most*** non-zero rows of the rrefs:

The Uniqueness of the Reduced Row Echelon Form: We are now in a position to prove that if A is an $m \times n$ matrix, and we obtain two matrices H and J from A using a finite sequence of elementary row operations, and both H and J are in reduced row echelon form, then $H = J$. Thus, the rref of A is ***unique***. We will use the Principle of Mathematical Induction.

1. a. First let us take care of the trivial case: If A consists entirely of zeroes, prove that $H = A = J$.
   Thus we can assume for the rest of the Exercise that A is a ***non-zero*** matrix.

   b. Explain why $rowspace(H) = rowspace(A) = rowspace(J)$.

   c. Explain why the number of non-zero rows of H must be the same as the number of non-zero rows of J. Hint: what does this number represent?

   Thus we can conclude that both H and J have k non-zero rows, for some positive number k. We must now show that every pair of corresponding rows are equal. We will start at with the last non-zero row because it has the most number of zeroes.
   We proceed with a numeric warm-up:

   d. Both H and J below have rank 3:

$$H = \begin{bmatrix} 1 & 0 & 0 & 6 & 7 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 1 & 5 & -3 \end{bmatrix}; \; J = \begin{bmatrix} 1 & 0 & -2 & 0 & 7 \\ 0 & 1 & 4 & -0 & 4 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}.$$

   Explain why the 3rd row of J ***cannot*** be expressed as a linear combination of the three rows of H. Hint: use the fact that the leading 1 is in the 4th column and every entry to its left is ***zero***.

   e. Now, explain in general that the leading 1 in row k of H must be in the ***same column*** as the leading 1 in row k of J. Hint: pick the matrix whose leading one in row k is further to the ***right***.

   f. Both H and J below have their leading 1 in row 3 in the same column:

$$H = \begin{bmatrix} 1 & 0 & 0 & 6 & 7 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 1 & 5 & -3 \end{bmatrix}; \; J = \begin{bmatrix} 1 & 0 & 0 & -2 & 9 \\ 0 & 1 & 0 & 3 & -6 \\ 0 & 0 & 1 & 5 & -2 \end{bmatrix}.$$

   Explain why the 3rd row of J ***cannot*** be expressed as a linear combination of the three rows of H, and similarly, the 3rd row of H ***cannot*** be expressed as a linear combination of the three rows of J. Hint: use the fact that the leading ones in rows 1 and 2 of J are above zeroes in row 3.

   g. Explain in general that row k of H must be ***exactly the same*** as row k of J.

   h. Now, let us focus on row $k - 1$. Both H and J below are in rref, both have rank 3, and their 3rd rows are the same:

$$H = \begin{bmatrix} 1 & 0 & -8 & 0 & 7 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}; \; J = \begin{bmatrix} 1 & 5 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}.$$

Explain why the 2nd row of J ***cannot*** be expressed as a linear combination of the ***three*** rows of H. Note that this includes possibly using the 3rd row of H.

i. In the same spirit as parts (e) and (g), explain in general why row $k-1$ of H must be ***exactly the same*** as row $k-1$ of J.

j. Notice that we are working our way up each matrix. Generalize your arguments above: show that if we already know that rows i to k of H and J have already been shown to be equal, then row $i-1$ of H and J must also be equal.

   Since we can continue in this fashion until we reach row 1, this completes the proof that all rows of H must be exactly the same as the corresponding row of J.

k. Epilogue: In part (d), we focused on the last non-zero row. Suppose we looked at the ***first*** rows instead. Both H and J below are in rref and have rank 3. Show that row 1 of J is a linear combination of the three rows of H.

$$H = \begin{bmatrix} 1 & 0 & -8 & 0 & -9 \\ 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}; \; J = \begin{bmatrix} 1 & 4 & 0 & -3 & 0 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Section 2.3

p. 186 The example should read: Suppose that:

$$T_1, T_2 \; : \; \mathbb{R}^3 \to \mathbb{R}^2, \quad \text{given by:}$$

$$T_1(\langle x, y, z\rangle) = \langle 3x - 2y + 5z, 2x + y - 3z\rangle, \quad \text{and} \ldots$$

(the $2nd$ component of T_1 should be $2x + y - 3z$, not $2x + 4y - 3z$).

Exercise 16. Replace $[proj_{\Pi}]$ with $[proj_L]$, where $L = Span(\{\vec{n}\})$, the normal line of Π.

Section 2.6, Exercise 27 (a) should refer to Exercise 16 instead of 13.

Section 3.1 p. 278

The final example should read: Suppose we let $V = \mathbb{R}^2$, but with addition defined by:

$$\langle x_1, y_1\rangle \oplus \langle x_2, y_2\rangle = \left\langle 2x_1 + 2x_2, 2y_1 + 2y_2 \right\rangle. \; . \; .$$

Section 3.2 p. 286

One coefficient in the Example is labelled k_4. It should be c_4.

Section 3.3 p. 298

The inequality sign should be reversed in the definition: In this case, we can also write: $|Y| > |X|$ and say that the cardinality of Y is ***strictly bigger*** than the cardinality of X.

Section 3.5 Exercises 3 and 4, part(b) for both: the right side of the equations should not be $z(x)$. It should be $\vec{0}_2$ and $\vec{0}_3$, respectively.

Section 3.7 Exercise 1. refers to Exercise 8 instead in Section 3.6.

Section 5.1 p. 454. The inverse permutation should be $\sigma^{-1} = (4, 6, 1, 5, 2, 3)$.