Logic, Sets, Proofs — Handout

Outline of Proof Techniques

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Proof Techniques

Adapted from *Book of Proof* by Richard Hammack, please consult the book for more details.

Basic Templates

Direct Proof

Proposition: If P, then Q.

Proof. Suppose P.

$\vdots$

Therefore, Q. $\square$

Contrapositive

Proposition: If P, then Q.

Proof. Suppose $\sim Q$.

$\vdots$

Therefore, $\sim P$. $\square$

Proof by Contradiction

Proposition: P.

Proof. Suppose $\sim P$.

$\vdots$

Therefore, C & $\sim C$,
a contradiction. $\square$

Proof by Contradiction

Proposition: If P, then Q.

Proof. Suppose P and $\sim Q$.

$\vdots$

Therefore, C & $\sim C$,
a contradiction. $\square$

If-and-only-if

Proposition: P if and only if Q.

Proof.
Prove $P \Rightarrow Q$ using direct, contrapositive, or contradiction.
Prove $Q \Rightarrow P$ using direct, contrapositive, or contradiction. $\square$

Multiple Equivalences (TFAE)

Proposition: The following are equivalent:

1. P_1
2. P_2
3. $\ldots$
4. P_k

Proof.
Prove $P_1 \Rightarrow P_2$ using direct, contrapositive, or contradiction.
Prove $P_2 \Rightarrow P_3$ using direct, contrapositive, or contradiction.
Prove $P_i \Rightarrow P_{i+1}$, for $i = 3, \ldots, k-1$, using direct, contrapositive, or contradiction.
Prove $P_k \Rightarrow P_1$ using direct, contrapositive, or contradiction. □

Mathematical Induction

Proposition: The statements S_n, for all $n \in \mathbb{N}$, are all true (i.e. $S_1, S_2, S_3, \ldots$ are all true).

Proof. (By Induction).
Base Case Prove that S_1 is true.
Inductive Case Prove that given any integer k, the statement S_k implies S_{k+1} is true.

More explicitly:

Inductive Hypothesis Let $k \in \mathbb{N}$ and suppose that S_k is true.

$\vdots$

Therefore, S_{k+1} is also true.

Therefore, by mathematical induction, it follows that S_n is true for all $n \in \mathbb{N}$. □

Proofs with Sets

Recall: basic set constructions

$$
\begin{aligned}
A &= \{x \mid P(x)\} \quad \text{“Set builder notation”} \\
A \subseteq B &= \{x \mid x \in A \implies x \in B\} \\
A \cup B &= \{x \mid x \in A \text{ or } x \in B\} \\
A \cap B &= \{x \mid x \in A \text{ and } x \in B\} \\
A - B &= \{x \mid x \in A \text{ and } x \notin B\} \\
A^c &= \{x \mid x \in U \text{ and } x \notin A\} \quad \text{Note: need notion of universal set } U
\end{aligned}
$$

Set Belonging

Proposition: $a \in A = \{x \mid P(x)\}$.

Proof. Show that $P(a)$ is true. □

Set Belonging

Proposition: $a \in \{x \in S \mid P(x)\}$.

Proof.
1. Verify that $a \in S$.
2. Show that $P(a)$ is true. □

Subset (Direct)

Proposition: $A \subseteq B$.

Proof. Suppose $a \in A$.

⋮

Therefore, $a \in B$.
Thus, $a \in A$ implies $a \in B$, so it follows that $A \subseteq B$. □

Subset (Contrapositive)

Proposition: $A \subseteq B$.

Proof. Suppose $a \notin B$.

⋮

Therefore, $a \notin A$.
Thus, $a \notin B$ implies $a \notin A$, so it follows that $A \subseteq B$. □

Set Equality

Proposition: $A = B$.

Proof.
1. Prove that $A \subseteq B$.
2. Prove that $B \subseteq A$.

Therefore, since $A \subseteq B$ and $B \subseteq A$, it follows that $A = B$. □

Proofs with Functions

One-to-one (Direct)

Proposition: $f : X \to Y$ is one-to-one (or injective).

Proof. Suppose $x, y \in X$ and $x \neq y$.

$\vdots$

Therefore, $f(x) \neq f(y)$. $\square$

One-to-one (Contrapositive)

Proposition: $f : X \to Y$ is one-to-one (or injective).

Proof. Suppose $x, y \in X$ and $f(x) = f(y)$.

$\vdots$

Therefore, $x = y$. $\square$

Onto

Proposition: $f : X \to Y$ is onto (or surjective).

Proof. Suppose $y \in Y$.
Prove there exits $x \in X$ for which $f(x) = y$. $\square$

Bijective

Proposition: $f : X \to Y$ is bijective.

Proof.
1. Prove that $f : X \to Y$ is one-to-one.
2. Prove that $f : X \to Y$ is onto.

Therefore, $f : X \to Y$ is bijective. $\square$

Inverse

Proposition: The inverse of $f : X \to Y$ exists.

Proof.
Prove that f is bijective.
Therefore, the inverse of f exists. $\square$

Cardinality

Proposition: $\text{card}(X) = \text{card}(Y)$.

Proof.
Find a function $f : X \to Y$ and prove it is bijective.
Therefore, X and Y have the same cardinality. $\square$

Additional Techniques

Uniqueness

Proposition: There's only one object with property A.

Proof. Suppose there are two objects x and y satisfying property A.
Prove $x = y$ or arrive at a contradiction. □

Existence

Proposition: There exists x such that $P(x)$ is true.

Proof. Find or construct an example of an x that makes $P(x)$ true. □

Identities

Proposition: $A = D$.

Proof. We start with $A = B$. We have

$$\begin{aligned} A &= B && \text{(by Assumption, Definition, or Theorem justifying } A = B\text{)} \\ &= C && \text{(by Assumption, Definition, or Theorem justifying } B = C\text{)} \\ &= D && \text{(by Assumption, Definition, or Theorem justifying } C = D\text{).} \end{aligned}$$

Therefore, $A = D$. □