1.6 Independent Sets versus Spanning Sets

The concepts of *Spans* and *independence* are two of the most important concepts in Linear Algebra.

We will see Theorems connecting Spans of sets of vectors, and linearly independent or dependent sets.

Equality of Spans

Theorem: Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \mathbb{R}^m$, and k_1, $k_2, \ldots, k_n \in \mathbb{R}$ a list of n ***non-zero scalars.*** Let us form a new set: $S^/ = \{k_1\vec{v}_1, k_2\vec{v}_2, \ldots, k_n\vec{v}_n\}$. Then: $Span(S) = Span(S^/)$.

$$\begin{aligned} & c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \\ = {} & c_1\frac{k_1}{k_1}\vec{v}_1 + c_2\frac{k_2}{k_2}\vec{v}_2 + \cdots + c_n\frac{k_n}{k_n}\vec{v}_n \\ = {} & \frac{c_1}{k_1}(k_1\vec{v}_1) + \frac{c_2}{k_2}(k_2\vec{v}_2) + \cdots + \frac{c_n}{k_n}(k_n\vec{v}_n), \end{aligned}$$

$$\begin{aligned} & c_1(k_1\vec{v}_1) + c_2(k_2\vec{v}_2) + \cdots + c_n(k_n\vec{v}_n) \\ = {} & (c_1k_1)\vec{v}_1 + (c_2k_2)\vec{v}_2 + \cdots + (c_nk_n)\vec{v}_n, \end{aligned}$$

Example:

$$S = \left\{ \begin{array}{c} \langle 3, -2, 5, 7, 4 \rangle, \langle 2, -5, 3, 6, 0 \rangle, \\ \langle -1, 0, 4, -3, 2 \rangle \end{array} \right\}$$

$$S' = \left\{ \begin{array}{c} \langle 6, -15, 9, 18, 0 \rangle, \langle -5, 0, 20, -15, 10 \rangle, \\ \langle -6, 4, -10, -14, -8 \rangle \end{array} \right\}$$

Is $Span(S) = Span(S')$?

Theorem — The Equality of Spans Theorem:

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and $S^{/} = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ be two sets of vectors from some Euclidean space $\mathbb{R}^k$. Then: $Span(S) = Span(S^{/})$ ***if and only if*** every $\vec{v}_i$ can be written as a linear combination of the $\vec{w}_1$ through $\vec{w}_m$, ***and*** every $\vec{w}_j$ can also be written as a linear combination of the $\vec{v}_1$ through $\vec{v}_n$.

Proof:

$(\Rightarrow)$ $Span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\})$ includes $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_n$ themselves.

$(\Leftarrow)$ Now, suppose that every $\vec{v}_i$ can be written as a linear combination of the $\vec{w}_1$ through $\vec{w}_m$, ***and*** every $\vec{w}_j$ can also be written as a linear combination of the $\vec{v}_1$ through $\vec{v}_n$.

Think of the linear combination:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

$$
\begin{aligned}
\vec{v}_1 &= a_{1,1}\vec{w}_1 + a_{1,2}\vec{w}_2 + \cdots + a_{1,m}\vec{w}_m, \\
\vec{v}_2 &= a_{2,1}\vec{w}_1 + a_{2,2}\vec{w}_2 + \cdots + a_{2,m}\vec{w}_m, \ldots\ldots \\
\vec{v}_n &= a_{n,1}\vec{w}_1 + a_{n,2}\vec{w}_2 + \cdots + a_{n,m}\vec{w}_m,
\end{aligned}
$$

$$
\begin{aligned}
& c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \\
= \ & c_1(a_{1,1}\vec{w}_1 + a_{1,2}\vec{w}_2 + \cdots + a_{1.m}\vec{w}_m) + \\
& c_2(a_{2,1}\vec{w}_1 + a_{2,2}\vec{w}_2 + \cdots + a_{2.m}\vec{w}_m) + \cdots + \\
& c_n(a_{n,1}\vec{w}_1 + a_{n,2}\vec{w}_2 + \cdots + a_{n.m}\vec{w}_m).
\end{aligned}
$$

$$
\begin{aligned}
& c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \\
= \ & c_1a_{1,1}\vec{w}_1 + c_1a_{1,2}\vec{w}_2 + \cdots + c_1a_{1.m}\vec{w}_m + \\
& c_2a_{2,1}\vec{w}_1 + c_2a_{2,2}\vec{w}_2 + \cdots + c_2a_{2.m}\vec{w}_m + \cdots + \\
& c_na_{n,1}\vec{w}_1 + c_na_{n,2}\vec{w}_2 + \cdots + c_na_{n.m}\vec{w}_m \\
= \ & (c_1a_{1,1} + c_2a_{2,1} + \cdots + c_na_{n,1})\vec{w}_1 + \\
& (c_1a_{1,2} + c_2a_{2,2} + \cdots + c_na_{n,2})\vec{w}_2 + \cdots + \\
& (c_1a_{1,m} + c_2a_{2,m} + \cdots + c_na_{n,m})\vec{w}_m.
\end{aligned}
$$

Example:

$$Span(\{\langle 3,-5,2,-4\rangle,\langle 2,-4,1,-2\rangle\})$$

$$vs.$$

$$Span(\langle 8,-14,5,-10\rangle,\langle -4,14,1,-2\rangle,\langle 1,3,3,-6\rangle).$$

Theorem — The Elimination Theorem:

Suppose that $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a ***linearly dependent*** set of vectors from $\mathbb{R}^m$, and $\vec{v}_n = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{n-1}\vec{v}_{n-1}$. Then:

$$Span(S) = Span(S - \{\vec{v}_n\}).$$

In other words, we can ***eliminate*** $\vec{v}_n$ from S and still maintain the ***same Span***.

More generally, if $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}_m$, where ***none*** of the coefficients in this dependence equation is 0, then:

$$Span(S) = Span(S - \{\vec{v}_i\}),$$

for all $i = 1..n$.

Example: Let:

$$S = \left\{ \begin{array}{c} \langle 3, 3, 5, 2, 4\rangle, \langle 3, 2, 0, -1, 6\rangle, \\ \langle 12, 12, 20, 8, 16\rangle, \langle 0, 1, 5, 3, -2\rangle \end{array} \right\},$$

and let us call these vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3,$ and $\vec{v}_4,$ in that order.

Theorem — The Minimizing Theorem:

Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a set of vectors from $\mathbb{R}^m$, and let $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix}$ be the $m \times n$ matrix with $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ as its ***columns***.

Suppose that R is the rref of A, and $i_1, i_2, \dots, i_k$ are the columns of R that contain the ***leading variables***. Then the set $S^/ = \{\vec{v}_{i_1}, \vec{v}_{i_2}, \dots, \vec{v}_{i_k}\}$, that is, the subset of vectors of S consisting of the corresponding columns of A, is a ***linearly independent*** set, and:

$$Span(S) = Span(S^/).$$

Furthermore, every $\vec{v}_i \in S - S^/$, that is, the vectors of S corresponding to the ***free variables*** of R, can be expressed as linear combinations of the vectors of $S^/$, using the ***coefficients*** found in the corresponding column of R.

Idea:

$$S = \left\{ \begin{array}{c} \langle 7, 4, -3, 11\rangle, \langle 2, -1, -1, 2\rangle, \langle 31, 22, -13, 51\rangle, \\ \langle 5, -2, 1, 5\rangle, \langle 17, 12, -21, 29\rangle \end{array} \right\}.$$

$$A = \begin{bmatrix} 7 & 2 & 31 & 5 & 17 \\ 4 & -1 & 22 & -2 & 12 \\ -3 & -1 & -13 & 1 & -21 \\ 11 & 2 & 51 & 5 & 29 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 5 & 0 & 3 \\ 0 & 1 & -2 & 0 & 8 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem — The Dependent vs. Spanning Sets Theorem:

Suppose we have a set of n vectors:

$$S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\},$$

from some Euclidean space $\mathbb{R}^k$, and we form $Span(S)$. Suppose now we randomly choose a set of m vectors from $Span(S)$ to form a new set:

$$L = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}.$$

We can now conclude that if $m > n$, then L is automatically linearly ***dependent***.

In other words, if we chose ***more*** vectors from $Span(S)$ than the number of vectors we used to ***generate*** S, then this new set will certainly be ***dependent***.

Proof:

$$\begin{aligned}\vec{u}_1 &= a_{1,1}\vec{w}_1 + a_{1,2}\vec{w}_2 + \cdots + a_{1,n}\vec{w}_n,\\ \vec{u}_2 &= a_{2,1}\vec{w}_1 + a_{2,2}\vec{w}_2 + \cdots + a_{2,n}\vec{w}_n, \;\ldots\ldots\\ \vec{u}_m &= a_{m,1}\vec{w}_1 + a_{m,2}\vec{w}_2 + \cdots + a_{m,n}\vec{w}_n.\end{aligned}$$

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_m\vec{u}_m = \vec{0}_k.$$

$$
\begin{aligned}
\vec{0}_k &= c_1(a_{1,1}\vec{w}_1 + a_{1,2}\vec{w}_2 + \cdots + a_{1,n}\vec{w}_n) + \\
&\quad c_2(a_{2,1}\vec{w}_1 + a_{2,2}\vec{w}_2 + \cdots + a_{2,n}\vec{w}_n) + \cdots + \\
&\quad c_n(a_{m,1}\vec{w}_1 + a_{m,2}\vec{w}_2 + \cdots + a_{m,n}\vec{w}_n) \\
&= c_1a_{1,1}\vec{w}_1 + c_1a_{1,2}\vec{w}_2 + \cdots + c_1a_{1,n}\vec{w}_n + \\
&\quad c_2a_{2,1}\vec{w}_1 + c_2a_{2,2}\vec{w}_2 + \cdots + c_2a_{2,n}\vec{w}_n + \cdots + \\
&\quad c_ma_{m,1}\vec{w}_1 + c_ma_{m,2}\vec{w}_2 + \cdots + c_ma_{m,n}\vec{w}_n \\
&= (c_1a_{1,1} + c_2a_{2,1} + \cdots + c_ma_{m,1})\vec{w}_1 + \\
&\quad (c_1a_{1,2} + c_2a_{2,2} + \cdots + c_ma_{m,2})\vec{w}_2 + \cdots + \\
&\quad (c_1a_{1,n} + c_2a_{2,n} + \cdots + c_ma_{m,n})\vec{w}_n.
\end{aligned}
$$

Now, we can ***force*** a solution if we set ***all*** of the coefficients of the vectors $\vec{w}_1$ through $\vec{w}_n$ to be zero:

$$c_1 a_{1,1} + c_2 a_{2,1} + \cdots + c_m a_{m,1} = 0,$$
$$c_1 a_{1,2} + c_2 a_{2,2} + \cdots + c_m a_{m,2} = 0, \;\ldots\ldots \;\text{ and}$$
$$c_1 a_{1,n} + c_2 a_{2,n} + \cdots + c_m a_{m,n} = 0.$$

Theorem — The Independent vs. Spanning Sets Theorem:

Suppose we have a set of n vectors $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ from some Euclidean space $\mathbb{R}^k$, and we form $Span(S)$.

Suppose now we randomly choose a set of m vectors from $Span(S)$ to form a new set:

$$L = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}.$$

We can now conclude that if L is ***independent***, then $m \leq n$.

Theorem — The Extension Theorem:

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a ***linearly independent*** set of vectors from $\mathbb{R}^m$, and suppose $\vec{v}_{n+1}$ is ***not*** a member of $Span(S)$. Then, the extended set:

$$\begin{aligned} S^{/} &= S \cup \{\vec{v}_{n+1}\} \\ &= \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{v}_{n+1}\} \end{aligned}$$

is ***still linearly independent***.