1.8 The Fundamental Matrix Spaces

Definitions/Theorem —
The Four Fundamental Matrix Spaces:

Let A be an $m \times n$ matrix. The ***rowspace*** of A is the Span of the rows of A. The ***columnspace*** of A is the Span of the columns of A. The ***nullspace*** of A is the set of all solutions to $A\vec{x} = \vec{0}_m$:

$$rowspace(A) = Span(\{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m\}),$$

$$colspace(A) = Span(\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}), \text{ and}$$

$$nullspace(A) = \left\{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}_m \right\},$$

where $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$ are the rows of A (considered as vectors from $\mathbb{R}^n$),
and $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$ are the columns of A (considered as vectors from $\mathbb{R}^m$).

Let us define the ***transpose*** matrix operation, where $A^\top$ (pronounced "A transpose") is the $n \times m$ matrix obtained from A by writing row 1 of A as column 1 of $A^\top$, writing row 2 of A as column 2 of $A^\top$, and so on.

The fourth fundamental matrix space is:

$$nullspace(A^\top) = \left\{ \vec{x} \in \mathbb{R}^m \mid A^\top \vec{x} = \vec{0}_n \right\},$$

Under these definitions, the subspaces and the corresponding ambient spaces are:

$$rowspace(A) = colspace(A^\top) \trianglelefteq \mathbb{R}^n,$$
$$colspace(A) = rowspace(A^\top) \trianglelefteq \mathbb{R}^m,$$
$$nullspace(A) \trianglelefteq \mathbb{R}^n, \text{ and}$$
$$nullspace(A^\top) \trianglelefteq \mathbb{R}^m.$$

The subspace properties for $nullspace(A)$:

zero vector?

closure under addition?

scalar multiplication?

Theorem — Basis for the Rowspace:

Elementary row operations do not change the rowspace of a matrix. Thus, if B is obtained from A using an elementary row operation, then $rowspace(A) = rowspace(B)$.
Consequently, if R is the rref of A, then the ***non-zero*** rows of R form a ***basis*** for rowspace(A).

Theorem — The Minimizing Theorem (Basis for Columnspace Version):

If an $m \times n$ matrix A has reduced row echelon form R, then the columns of A that correspond to the leading columns of R form a ***basis*** for the ***columnspace*** of A.

Theorem — Basis for Nullspace:

Let A be an $m \times n$ matrix with rref R. Then:

$$nullspace(A) = nullspace(R).$$

Furthermore, if R has k free variables, then $nullspace(A)$ will be k-dimensional, and we obtain a basis for $nullspace(A)$ by solving for the leading variables in terms of the free variables, as usual. A similar equation applies to $A^\top$ and its rref.

Warning: We can ***directly*** use the entries of the rref of A to find a basis only for the ***rowspace*** and ***nullspace*** of A. However, we have to go back to the ***original*** columns of A to describe the ***columnspace*** of A, using the leading 1's as our guides.

Example: Suppose we have the matrix:

$$A = \begin{bmatrix} 7 & -28 & 2 & 17 & -3 & 73 & 24 \\ -3 & 12 & 4 & -17 & 2 & -3 & -22 \\ -1 & 4 & 24 & -51 & 4 & 131 & -62 \\ 2 & -8 & -3 & 12 & 4 & -43 & 37 \end{bmatrix}$$

with rref:

$$R = \begin{bmatrix} 1 & -4 & 0 & 3 & 0 & 5 & 6 \\ 0 & 0 & 1 & -2 & 0 & 7 & -3 \\ 0 & 0 & 0 & 0 & 1 & -8 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$A^\top = \begin{bmatrix} 7 & -3 & -1 & 2 \\ -28 & 12 & 4 & -8 \\ 2 & 4 & 24 & -3 \\ 17 & -17 & -51 & 12 \\ -3 & 2 & 4 & 4 \\ 73 & -3 & 131 & -43 \\ 24 & -22 & -62 & 37 \end{bmatrix}$$

with rref:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank and Nullity

Definition/Theorem: Rank and Nullity:

Let A be an $m \times n$ matrix. The dimension of the ***nullspace*** of A is called the ***nullity*** of A.

The dimension of the ***rowspace*** of A is exactly the same as the dimension of the ***columnspace*** of A, and we call this common dimension the ***rank*** of A.

Furthermore, since $rowspace(A) = colspace(A^\top)$, and $colspace(A) = rowspace(A^\top)$, we can conclude that $rank(A) = rank(A^\top)$.

We write these dimensions symbolically as:

$$\begin{aligned} rank(A) &= dim(rowspace(A)) \\ &= dim(colspace(A)) = rank(A^\top), \\ nullity(A) &= dim(nullspace(A)), \quad \text{and} \\ nullity(A^\top) &= dim(nullspace(A)). \end{aligned}$$

Example: For the matrix in our previous Example:

$$\begin{aligned} rank(A) &= \\ nullity(A) &= \\ nullity(A^\top) &= \end{aligned}$$

Theorem/Definition — Bounds on Rank and Nullity:

Suppose A is an $m \times n$ matrix. Then:

$$0 \leq rank(A) = rank(A^\top) \leq \min(m,n),$$

$$n - m \leq nullity(A) \leq n, \quad \text{and}$$

$$m - n \leq nullity(A^\top) \leq m.$$

We say that A has ***full-rank*** if $rank(A) = \min(m,n)$.

The Dimension Theorem for Matrices

Theorem — The Dimension Theorem for Matrices:

For any $m \times n$ matrix A:

$$rank(A) + nullity(A) = n, \text{ and similarly,}$$

$$rank(A^\top) + nullity(A^\top) = m.$$

Sight-Reading the Nullspace

Note how a column of numbers turns into the components of each basis vector for the nullspace, but appear with the ***opposite*** sign.

Where does each component go?

The General Solution of $A\vec{x} = \vec{b}$

Theorem — The Columnspace Test for Consistency:

The matrix equation $A\vec{x} = \vec{b}$ is ***consistent*** if and only if $\vec{b} \in colspace(A)$.

Furthermore, if $A\vec{x} = \vec{b}$ is consistent, suppose $\vec{x}_p$ is a ***fixed*** solution (also called a ***particular*** solution) of this system. Then, a vector $\vec{x}$ is a solution of this system ***if and only if*** it can be written in the form: $\vec{x} = \vec{x}_p + \vec{x}_h$, where $\vec{x}_h$ is a member of the $\boldsymbol{nullspace}(A)$.

Consequently, if $\vec{x}$ and $\vec{y}$ are any two solutions to $A\vec{x} = \vec{b}$, then $\vec{x} - \vec{y} \in nullspace(A)$.

Definition: If $\vec{b}$ is a fixed vector of $\mathbb{R}^n$, and $W \trianglelefteq \mathbb{R}^n$, then: $\vec{b} + W = \left\{\vec{b} + \vec{w} \mid \vec{w} \in W\right\}$ is called a ***translate*** of the subspace W.

Theorem: The set X of all solutions $\vec{x}$ of a consistent matrix equation of $A\vec{x} = \vec{b}$ is a translate of the nullspace, that is:

$$X = \vec{x}_p + nullspace(A),$$

where $\vec{x}_p$ is a fixed or ***particular*** solution for $A\vec{x} = \vec{b}$.

Example:

$$\left[A\,|\,\vec{b}\right] = \left[\begin{array}{ccccc|c} 3 & -15 & -5 & 1 & 3 & 2 \\ -2 & 10 & 3 & -2 & -2 & -3 \\ 4 & -20 & -5 & 8 & 3 & 5 \\ 2 & -10 & -4 & -2 & 2 & -2 \end{array}\right],$$

with rref:

$$R = \left[\begin{array}{ccccc|c} \mathbf{1} & -5 & 0 & 7 & 0 & 3 \\ 0 & 0 & \mathbf{1} & 4 & 0 & 5 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Properties of Full-Rank Matrices

Theorem — Linear Systems with a Full-Rank Coefficient Matrix: Suppose that $\left[A|\vec{b}\right]$ is an augment matrix, where A is an $m \times n$ ***full-rank*** matrix. Then:

1. If $m < n$ (the system is ***underdetermined***), then the system is ***consistent*** for ***any*** $\vec{b} \in \mathbb{R}^m$, and furthermore, the system always has an ***infinite*** number of solutions.

2. If $m = n$ (the system is ***square***), then the system is ***consistent*** for ***any*** $\vec{b} \in \mathbb{R}^m$, and furthermore, the system has ***exactly one*** solution.

3. If $m > n$ (the system is *overdetermined*), ***and*** the system is ***consistent***, then it has ***exactly one*** solution. However, there is ***at least one*** $\vec{b} \in \mathbb{R}^m$ for which the system is ***inconsistent***.

Thus, we can also say that an overdetermined full-rank system has ***at most one*** solution.

Example: Consider:

$$A_1 = \begin{bmatrix} -3 & -5 & -6 & 2 \\ 2 & 6 & -4 & 1 \\ 4 & 7 & 7 & -5 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -3 & -5 & 2 \\ 2 & 6 & -3 \\ 4 & 7 & -5 \end{bmatrix}, \text{ and}$$

$$A_3 = \begin{bmatrix} 3 & 5 & -2 \\ -2 & 0 & 4 \\ 1 & -3 & -3 \\ 5 & 6 & -5 \end{bmatrix}.$$

Study the systems:

$$A_1\vec{x} = \begin{bmatrix} -1 \\ -4 \\ 5 \end{bmatrix},$$

$$A_2\vec{y} = \begin{bmatrix} -1 \\ -4 \\ 5 \end{bmatrix}, \text{ and}$$

$$A_3\vec{z} = \begin{bmatrix} -1 \\ 4 \\ 2 \\ -3 \end{bmatrix}.$$

$$\left[\begin{array}{cccc|c} -3 & -5 & -6 & 2 & -1 \\ 2 & 6 & -4 & -3 & -4 \\ 4 & 7 & 7 & -5 & 5 \end{array}\right]$$

with rref

$$R_1 = \begin{bmatrix} 1 & 0 & 7 & 0 & 4 \\ 0 & 1 & -3 & 0 & -3 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} -3 & -5 & 2 & -1 \\ 2 & 6 & -3 & -4 \\ 4 & 7 & -5 & 5 \end{array}\right]$$

with rref

$$R_2 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -2 & -1 \\ -2 & 0 & 4 & 4 \\ 1 & -3 & -3 & 2 \\ 5 & 6 & -5 & -3 \end{array}\right]$$

with rref

$$R_3 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -2 & -1 \\ -2 & 0 & 4 & \boxed{5} \\ 1 & -3 & -3 & 2 \\ 5 & 6 & -5 & -3 \end{array}\right]$$

with rref

$$R_4 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$