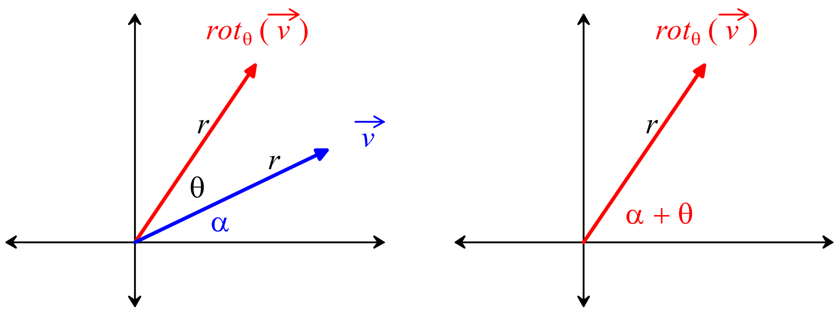
2.2 Rotations, Projections and Reflections

Rotations in $\mathbb{R}^2$



A Vector $\vec{v}$ and $rot_\theta(\vec{v})$,

its Counterclockwise Rotation by θ

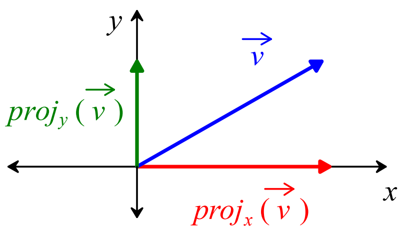
Theorem: The function $rot_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ that takes a vector $\vec{v}$ and rotates $\vec{v}$ counterclockwise by an angle of θ about the origin is a ***linear transformation***, with:

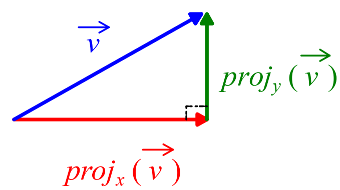
$$[rot_\theta] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Basic Projections in $\mathbb{R}^2$

$$proj_x(\langle x,y\rangle) = \langle x,0\rangle.$$

$$proj_y(\langle x,y\rangle) = \langle 0,y\rangle.$$





Key Relationship:

$$\begin{aligned}\vec{v} &= \langle x,y\rangle \\ &= \langle x,0\rangle + \langle 0,y\rangle \\ &= proj_x(\vec{v}) + proj_y(\vec{v})\end{aligned}$$

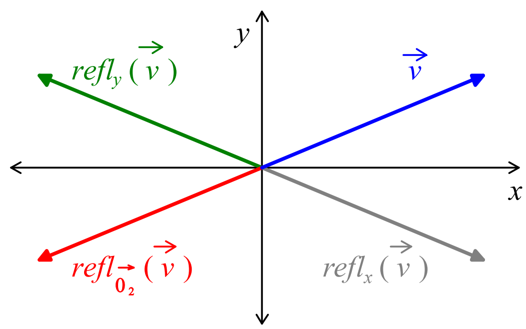
$$proj_x(\vec{v}) = \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ and}$$

$$proj_y(\vec{v}) = \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[proj_x] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[proj_y] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Basic Reflections in $\mathbb{R}^2$

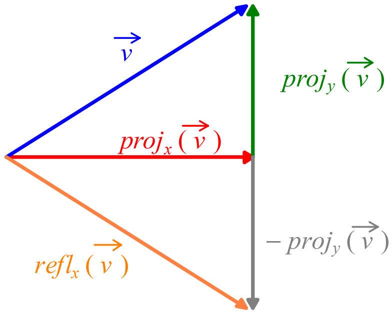


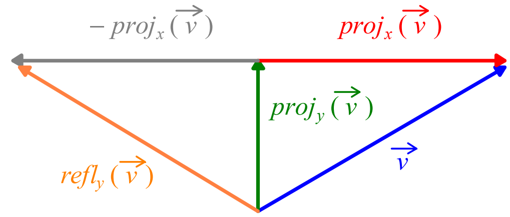
A Vector $\vec{v}$ and its Three Basic Reflections in $\mathbb{R}^2$.

$$\mathit{refl}_x\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

$$\mathit{refl}_y\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and}$$

$$\mathit{refl}_{\vec{0}_2}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

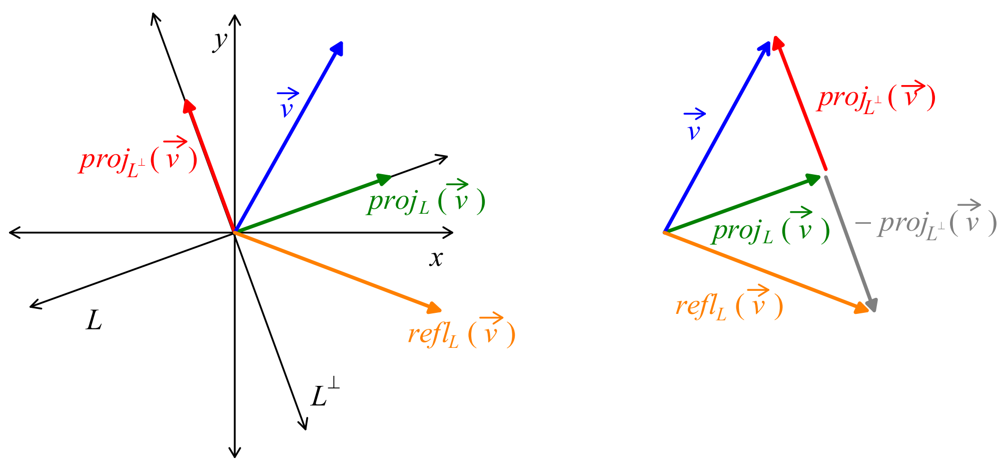




The Geometric Relationships Among $\vec{v}$, $proj_x(\vec{v})$, $proj_y(\vec{v})$, $refl_x(\vec{v})$ and $refl_y(\vec{v})$

$$refl_x(\vec{v}) = proj_x(\vec{v}) - proj_y(\vec{v}), \text{ and}$$
$$refl_y(\vec{v}) = proj_y(\vec{v}) - proj_x(\vec{v}).$$

General Projections and Reflections in $\mathbb{R}^2$



The Projections of $\vec{v}$ Onto a Line L and its Orthogonal Complement $L^\perp$, and the Reflection of $\vec{v}$ Across L.

Basic Projections in $\mathbb{R}^3$

$$proj_x(\langle x, y, z\rangle) = \langle x, 0, 0\rangle$$

$$proj_y(\langle x, y, z\rangle) = \langle 0, y, 0\rangle$$

$$proj_z(\langle x, y, z\rangle) = \langle 0, 0, z\rangle$$

$$proj_{xy}(\langle x, y, z\rangle) = \langle x, y, 0\rangle$$

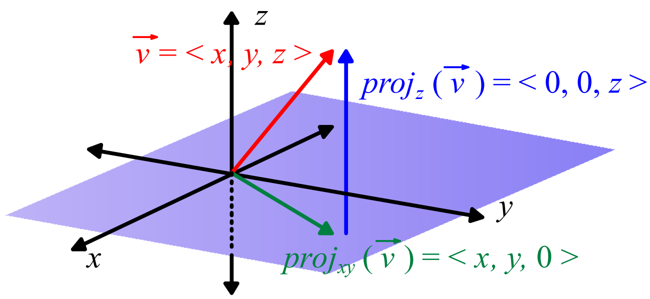
$$proj_{xz}(\langle x, y, z\rangle) = \langle x, 0, z\rangle$$

$$proj_{yz}(\langle x, y, z\rangle) = \langle 0, y, z\rangle$$

$$
\begin{aligned}
\vec{v} &= \langle x, y, z \rangle \\
&= \langle x, 0, 0 \rangle + \langle 0, y, z \rangle \\
&= proj_x(\langle x, y, z \rangle) + proj_{yz}(\langle x, y, z \rangle),
\end{aligned}
$$

$$
\begin{aligned}
\vec{v} &= \langle x, y, z \rangle \\
&= \langle 0, y, 0 \rangle + \langle x, 0, z \rangle \\
&= proj_y(\langle x, y, z \rangle) + proj_{xz}(\langle x, y, z \rangle), \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
\vec{v} &= \langle x, y, z \rangle \\
&= \langle 0, 0, z \rangle + \langle x, y, 0 \rangle \\
&= proj_z(\langle x, y, z \rangle) + proj_{xy}(\langle x, y, z \rangle)
\end{aligned}
$$



The Relationships Among $\vec{v}$, $proj_{xy}(\vec{v})$ and $proj_z(\vec{v})$.

$$\begin{aligned} refl_{xy}(\langle x,y,z\rangle) &= \langle x,y,-z\rangle \\ &= \langle x,y,0\rangle - \langle 0,0,z\rangle \\ &= proj_{xy}(\langle x,y,z\rangle) - proj_z(\langle x,y,z\rangle). \\ refl_z(\langle x,y,z\rangle) &= proj_z(\langle x,y,z\rangle) - proj_{xy}(\langle x,y,z\rangle) \\ &= \langle 0,0,z\rangle - \langle x,y,0\rangle \\ &= \langle -x,-y,z\rangle. \end{aligned}$$

The Basic ***Reflection Operators***:

$$refl_x(\langle x, y, z\rangle) = \langle x, -y, -z\rangle$$

$$refl_y(\langle x, y, z\rangle) = \langle -x, y, -z\rangle$$

$$refl_z(\langle x, y, z\rangle) = \langle -x, -y, z\rangle$$

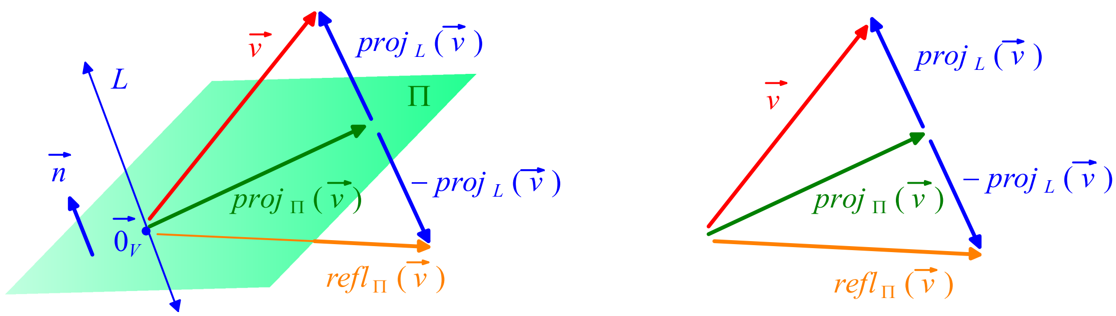
$$refl_{xy}(\langle x, y, z\rangle) = \langle x, y, -z\rangle$$

$$refl_{xz}(\langle x, y, z\rangle) = \langle x, -y, z\rangle$$

$$refl_{yz}(\langle x, y, z\rangle) = \langle -x, y, z\rangle$$

$$refl_{\vec{0}_3}(\langle x, y, z\rangle) = \langle -x, -y, -z\rangle$$

General Projections and Reflections in $\mathbb{R}^3$



We need the decomposition:

$$\vec{v} = proj_\Pi(\vec{v}) + proj_L(\vec{v}),$$

where $proj_\Pi(\vec{v}) \in \Pi$ and $proj_L(\vec{v}) \in L$.

General Principle:

$$\text{If } \vec{v} = proj_W(\vec{v}) + proj_{W^\perp}(\vec{v}), \text{ then:}$$

$$refl_W(\vec{v}) = proj_W(\vec{v}) - proj_{W^\perp}(\vec{v})$$

For planes Π and lines L through the origin:

$$refl_\Pi(\vec{v}) = proj_\Pi(\vec{v}) - proj_L(\vec{v}), \text{ and}$$

$$refl_L(\vec{v}) = proj_L(\vec{v}) - proj_\Pi(\vec{v}).$$