2.4 Properties of Operations on Linear Transformations and Matrices

Goal: Show that matrix operations enjoy many (but not all!!!) of the properties of the analogous operations on ordinary real numbers.

Properties of Matrix Addition and Scalar Multiplication

Theorem: If A, B and C are $m \times n$ matrices, and r and s are scalars, then the following properties hold:

1. *The Commutative Property of Addition:*

$$A + B = B + A$$

2. *The Associative Property of Addition:*

$$A + (B + C) = (A + B) + C$$

3. *The "Left" Distributive Property:*

$$(r + s)A = rA + sA$$

4. *The "Right" Distributive Property:*

$$r(A + B) = rA + rB$$

5. *The Associative Property of Scalar Multiplication:*

$$r(sA) = (rs)A = s(rA)$$

Properties of Matrix Multiplication

Theorem: If A and B are $m \times k$ matrices, C and D are $k \times n$ matrices, and r is a scalar, then the following properties hold:

1. ***The "Left" Distributive Property:***

$$(A + B)C = AC + BC$$

2. ***The "Right" Distributive Property:***

$$A(C + D) = AC + AD$$

3. ***The Associative Property of Mixed (Scalar and Matrix) Products:***

$$r(BC) = (rB)C = B(rC)$$

The Associative Property of Matrix Multiplication

Theorem: If A is an $m \times p$ matrix, B is a $p \times q$ matrix, and C is a $q \times n$ matrix, then $A(BC) = (AB)C$.

Proof:

Both products $A(BC)$ and $(AB)C$ are $m \times n$ matrices.

Now, we have to show that both sides, pair-wise, have exactly the ***same entries***.

Case 1: $C = \vec{x}$, a $q \times 1$ matrix.

$$B = \left[\vec{b}_1 \; \vec{b}_2 \; \ldots \; \vec{b}_q \right]$$

$$AB = \left[A\vec{b}_1 \; A\vec{b}_2 \; \ldots \; A\vec{b}_q \right]$$

$$(AB)\vec{x} = \left[A\vec{b}_1\ A\vec{b}_2\ \dots\ A\vec{b}_q\right]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{bmatrix}$$

$$= x_1\left(A\vec{b}_1\right) + x_2\left(A\vec{b}_2\right) + \cdots + x_q\left(A\vec{b}_q\right)$$

Now, let us work on $A(B\vec{x})$:

$$B\vec{x} = \left[\vec{b}_1\ \vec{b}_2\ \dots\ \vec{b}_q\right]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{bmatrix}$$

$$= x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_q\vec{b}_q$$

$$
\begin{aligned}
A(B\vec{x}) &= A\left(x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_q\vec{b}_q\right) \\
&= A\left(x_1\vec{b}_1\right) + A\left(x_2\vec{b}_2\right) + \cdots + A\left(x_q\vec{b}_q\right) \\
&\quad \text{(by the “Right” Distributive Property)} \\
&= x_1\left(A\vec{b}_1\right) + x_2\left(A\vec{b}_2\right) + \cdots + x_q\left(A\vec{b}_q\right)
\end{aligned}
$$

Case 2: C is an arbitrary $q \times n$ matrix:

$$C = \left[\vec{c}_1 \;\; \vec{c}_2 \; \ldots \; \vec{c}_n\right]$$

$$(AB)\vec{c}_i = A(B\vec{c}_i)$$

for every column $\vec{c}_i$.

Thus, column i of $(AB)C$ is exactly the same as that of $A(BC)$, and therefore $(AB)C = A(BC)$.

The Matrix of a Composition

Theorem: If $T_1 : \mathbb{R}^n \to \mathbb{R}^k$ and $T_2 : \mathbb{R}^k \to \mathbb{R}^m$ are linear transformations, then:

$$[T_2 \circ T_1] = [T_2][T_1]$$

k-fold Compositions

If $T_1, T_2, \ldots, T_{k-1}, T_k$ are all linear transformations with the property that ***the codomain of*** T_i ***is the domain of*** T_{i+1}, for all $i = 1..k-1$, then we can inductively construct the k***–fold composition*** of these linear transformations by:

$$\begin{aligned}&(T_k \circ T_{k-1} \circ \cdots \circ T_2 \circ T_1)(\vec{v})\\ = \; &T_k((T_{k-1} \circ \cdots \circ T_2 \circ T_1)(\vec{v}))\end{aligned}$$

$$[T_k \circ T_{k-1} \circ \cdots \circ T_2 \circ T_1] = [T_k][T_{k-1}]\cdots[T_2][T_1]$$

Powers of Square Matrices and Linear Operators

Theorem: The matrix product AA can be formed ***if and only if*** A is an $n \times n$ matrix. Analogously, the composition $T \circ T$ can be formed if and only if $T : \mathbb{R}^n \to \mathbb{R}^n$, i.e., T is an ***operator***.

Write AA as A^2 and $T \circ T$ as T^2.

Similarly, by induction, we will write:

$$A^k = A \bullet A^{k-1} = A \bullet A \bullet \cdots \bullet A, \text{ and}$$

$$T^k(\vec{v}) = T(T^{k-1}(\vec{v})) = T(T(\ldots T(\vec{v})))$$

Evaluating a Polynomial with a Matrix:

Definition: If $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k$ is a polynomial with real coefficients, and A is any $n \times n$ matrix, then we define the ***polynomial evaluation***, $p(A)$, by:

$$p(A) = c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_k A^k.$$

Multiplication by I_n *and* $0_{m\times n}$

Theorem: If A is any $m \times n$ matrix, then:

$$AI_n = A \qquad \text{and} \qquad I_mA = A;$$

$$A0_{n\times p} = 0_{m\times p} \quad \text{and} \quad 0_{q\times m}A = 0_{q\times n}.$$

Danger Zone!

The Existence of Zero Divisors:

Definition: Two $n \times n$ matrices A and B with the property that $AB = 0_{n\times n}$, but ***neither*** A nor B is $0_{n\times n}$ are called ***zero divisors***.

In other words, The Zero Factors Theorem does not hold for matrices.

$AB \neq BA$ *Most of the Time!*

Matrix multiplication, in general, is NOT commutative!

A Linear Transformation is Uniquely Determined by any Basis

Theorem: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is any ***basis*** for $\mathbb{R}^n$, then the action of T is uniquely determined by the vectors $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)\}$ from $\mathbb{R}^m$.

More specifically, if $\vec{v} \in \mathbb{R}^n$ and $\vec{v}$ is expressed (uniquely) as $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$, then:

$$T(\vec{v}) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_n T(\vec{v}_n).$$