2.6 Invertible Operators and Matrices

In Algebra, we first require that a function:

$$f : D \to R,$$

with domain D and range R, is ***one-to-one*** on D before we find its inverse. If so:

$$f^{-1} : R \to D, \text{ where}$$

$$f^{-1}(y) = x \quad \textbf{\textit{if and only if}} \quad f(x) = y.$$

f and f^{-1} also possess the ***cancellation properties***:

$$f^{-1}(f(x)) = f^{-1}(y) = x \text{ for all } x \in D, \quad \textbf{\textit{and}}$$
$$f(f^{-1}(y)) = f(x) = y \text{ for all } y \in R.$$

Definition: We say that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***invertible*** if and only if T is ***both*** one-to-one and onto.

We also say equivalently that T is ***bijective***, T is a ***bijection*** or T is an ***isomorphism***.

Theorem: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is invertible, then $n = m$.

Theorem: A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible ***if and only if*** we can find another ***unique*** linear operator, $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, the ***inverse operator*** for T, such that if $\vec{v} \in \mathbb{R}^n$ and $T(\vec{v}) = \vec{w}$, then we define:

$$T^{-1}(\vec{w}) = \vec{v},$$

and thus:

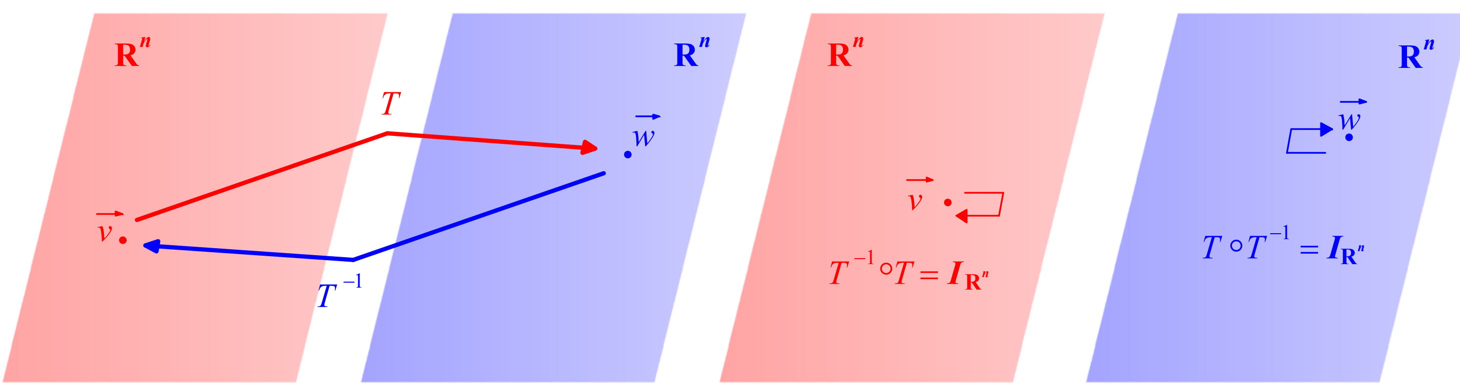
$$(T^{-1} \circ T)(\vec{v}) = \vec{v} \quad \text{and} \quad (T \circ T^{-1})(\vec{w}) = \vec{w}.$$

In other words:

$$T^{-1} \circ T = T \circ T^{-1} = I_{\mathbb{R}^n},$$

the ***identity operator*** on $\mathbb{R}^n$.

Furthermore, if T is invertible, then T^{-1} is also invertible, and $(T^{-1})^{-1} = T$. Thus, we can say that T and T^{-1} are ***inverses of each other***.



The Composition of T with T^{-1}

$$T^{-1} \circ T = I_{\mathbb{R}^n} = T \circ T^{-1}$$

Invertible Matrices

Definition: An $n \times n$ matrix A is ***invertible*** if and only if the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ corresponding to $A = [T]$ is an invertible operator. In other words, the operator defined by:

$$T(\vec{v}) = A\vec{v},$$

for all $\vec{v} \in \mathbb{R}^n,$ is an invertible operator on $\mathbb{R}^n$.

Theorem/Definition: An $n \times n$ matrix A is invertible ***if and only if*** we can find another $n \times n$ matrix B such that:

$$AB = I_n = BA.$$

We call B the ***inverse matrix*** of A, and denote it by A^{-1}.

If A is invertible, then the inverse matrix A^{-1} is likewise invertible, and:

$$(A^{-1})^{-1} = A.$$

In other words, $B^{-1} = A$. Thus, we can say that A and A^{-1} are ***inverses of each other***.

Theorem: If an $n \times n$ matrix A is invertible, then its inverse matrix B is ***unique***. This means that if B and C both satisfy the equations:

$$AB = I_n = BA \quad \text{and} \quad AC = I_n = CA,$$

then $B = C$.

Theorem: Suppose that:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then A is invertible ***if and only if*** $ad - bc \neq 0,$ in which case:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example:

$$A = \begin{bmatrix} -3 & -5 \\ 5 & 7 \end{bmatrix}.$$

Example:

$$A = \begin{bmatrix} 3 & -7 \\ 12 & -28 \end{bmatrix}.$$

The Matrix of T^{-1}

Theorem: A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible ***if and only if*** $A = [T]$ is an invertible $n \times n$ matrix. If this is the case, then:

$$[T^{-1}] = A^{-1} = [T]^{-1}.$$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by:

$$T(\langle x, y \rangle) = \langle 3x + 7y, 2x - 6y \rangle.$$

Bonus Example:

The Matrix of a Reflection Across a Plane in $\mathbb{R}^3$

Let Π be the plane in $\mathbb{R}^3$ with Cartesian equation:

$$3x - 5y + 2z = 0.$$

We found in the last Example of Section 2.2 that:

$$[refl_{\Pi}] = \begin{bmatrix} \frac{10}{19} & \frac{15}{19} & -\frac{6}{19} \\ \frac{15}{19} & -\frac{6}{19} & \frac{10}{19} \\ -\frac{6}{19} & \frac{10}{19} & \frac{15}{19} \end{bmatrix}$$