2.7 Finding the Inverse of a Matrix

Goal: to be able to construct the matrix of the inverse of an invertible linear operator, and at the same time, to find the inverse of an invertible square matrix which is 3×3 or bigger, when it is possible to do so.

Multiplicative Properties of Elementary Matrices

Theorem: If E is an elementary $n \times n$ matrix and A is any $n \times m$ matrix, then the ***matrix product*** EA can be computed by simply performing the ***same elementary row operation*** on A that was used to produce E from I_n.

An elementary matrix ***encodes*** the elementary row operation that produced it.

Example: Suppose that

$$A = \begin{bmatrix} 5 & 7 & -2 & 3 \\ 4 & 1 & 8 & -5 \\ 2 & -3 & 9 & 6 \end{bmatrix}$$

and:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

Then:

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & -2 & 3 \\ 4 & 1 & 8 & -5 \\ 2 & -3 & 9 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{bmatrix},$$

$$E_2A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 7 & -2 & 3 \\ 4 & 1 & 8 & -5 \\ 2 & -3 & 9 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{bmatrix}, \text{ and}$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & -2 & 3 \\ 4 & 1 & 8 & -5 \\ 2 & -3 & 9 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}$$

Theorem: Elementary matrices are ***invertible***, and the inverse of an elementary matrix is another elementary matrix of exactly the ***same type***.

Examples:

$$\text{For } E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1^{-1} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}.$$

$$\text{For } E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}.$$

$$\text{For } E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}.$$

A Preliminary Test for Invertibility

Theorem: Let A be an $n \times n$ matrix. Then A is invertible ***if and only if*** the rref of A is I_n.

A Method to Find A^{-1}

Theorem: Let A be an $n \times n$ matrix. If we construct the $n \times 2n$ augmented matrix:

$$\left[A \mid I_n \right],$$

then A is invertible ***if and only if*** the rref of this augmented matrix contains I_n in the first n columns. If this is the case, then A^{-1} will be found in the last n columns. In other words, the rref of $\left[A \mid I_n \right]$ is:

$$\left[I_n \mid A^{-1} \right]$$

Key Idea: there are only two possibilities for the rref of a square matrix.

Factoring Invertible Matrices

Theorem: An $n \times n$ matrix A is invertible ***if and only if*** it can be expressed as a product of elementary matrices. If this is the case, then more precisely, we can factor A as:

$$A = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1},$$

where $E_1, E_2, \ldots, E_k$ are the elementary matrices corresponding to a choice of elementary row operations we used in the Gauss-Jordan Algorithm to transform A into I_n.

Note: The factorization of A into elementary matrices is ***not unique***, since a different choice of elementary row operations will result in a different factorization.

Solving Invertible Square Equations

Theorem: If A is an invertible $n \times n$ matrix, then the system:

$$A\vec{x} = \vec{b}$$

has exactly one solution for any $n \times 1$ matrix $\vec{b}$, namely:

$$\vec{x} = A^{-1}\vec{b}.$$

More generally, if C is any $n \times m$ matrix, then the matrix equation:

$$AB = C$$

has exactly one solution for the $n \times m$ matrix B, namely:

$$B = A^{-1}C$$