3.1 Axioms for a Vector Space

Definition — The Axioms of an Abstract Vector Space:

A vector space $(V, \oplus, \odot)$ is a non-empty set V, together with two operations:

$$\oplus \quad \text{(vector addition), and}$$

$$\odot \quad \text{(scalar multiplication),}$$

such that: for all $\vec{u}$, $\vec{v}$ and $\vec{w} \in V$ and all r, $s \in \mathbb{R}$, $(V, \oplus, \odot)$ satisfies the following ten properties:

1. ***The Closure Property of Vector Addition:***

$$\vec{u} \oplus \vec{v} \in V$$

2. ***The Closure Property of Scalar Multiplication:***

$$r \odot \vec{u} \in V$$

3. *The Commutative Property of Vector Addition:*

$$\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$$

4. *The Associative Property of Vector Addition:*

$$(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$$

5. *The Existence of a Zero Vector:*

There exists $\overrightarrow{0}_V \in V$, such that: $\overrightarrow{0}_V \oplus \vec{v} = \vec{v} = \vec{v} \oplus \overrightarrow{0}_V$

6. *The Existence of Additive Inverses:*

There exists $-\vec{v} \in V$ such that: $\vec{v} \oplus (-\vec{v}) = \overrightarrow{0}_V = (-\vec{v}) \oplus \vec{v}$

7. *The Distributive Property of Ordinary Addition over Scalar Multiplication:*

$$(r + s) \odot \vec{v} = (r \odot \vec{v}) \oplus (s \odot \vec{v})$$

8. *The Distributive Property of Vector Addition over Scalar Multiplication:*

$$r \odot (\vec{u} \oplus \vec{v}) = (r \odot \vec{u}) \oplus (r \odot \vec{v})$$

9. *The Associative Property of Scalar Multiplication:*

$$r \odot (s \odot \vec{v}) = s \odot (r \odot \vec{v}) = (rs) \odot \vec{v}$$

10. *The Unitary Property of Scalar Multiplication:*

$$1 \odot \vec{v} = \vec{v}$$

We need ***three objects,*** that is, three pieces of ***information*** to define a vector space:

(1) a non-empty ***set*** V,

(***what*** are the vectors)

(2) a rule for ***vector addition*** $\oplus$ that tells us ***how to add*** two vectors to get another vector, and

(3) a rule for ***scalar multiplication*** $\odot$ that tells us ***how to multiply*** a real number with a vector to get another vector.

Polynomial Spaces

$$\mathbb{P}^n = \{p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}\}$$

Example: $\mathbb{P}^2$

$$p(x) = 3 - 5x + 7x^2 \quad \text{and}$$
$$q(x) = 4 - 3x^2 \in \mathbb{P}^2$$

$$\begin{aligned} p(x) \oplus q(x) &= (3 - 5x + 7x^2) + (4 - 3x^2) \\ &= 7 - 5x + 4x^2, \quad \text{and} \\ 3 \odot p(x) &= 3(3 - 5x + 7x^2) \\ &= 9 - 15x + 21x^2 \end{aligned}$$

$$\overrightarrow{0}_{\mathbb{P}^n} = z(x) = 0 + 0x + \cdots + 0x^n$$

$$-p(x) = -a_0 - a_1 x - a_2 x^2 - \cdots - a_n x^n$$

Functions Spaces

$$F(I) = \{ f(x) \mid f(a) \text{ is defined for all } a \in I \}$$

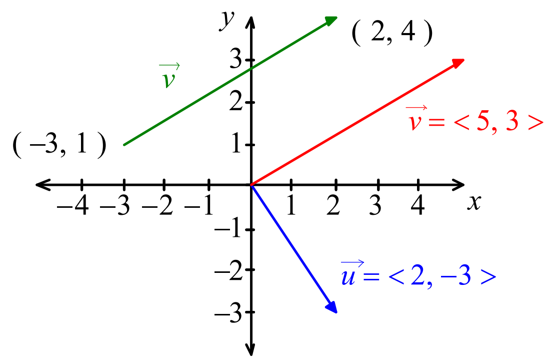
$$(f+g)(x) = f(x) + g(x), \quad \text{and}$$

$$(kf)(x) = k \cdot f(x)$$

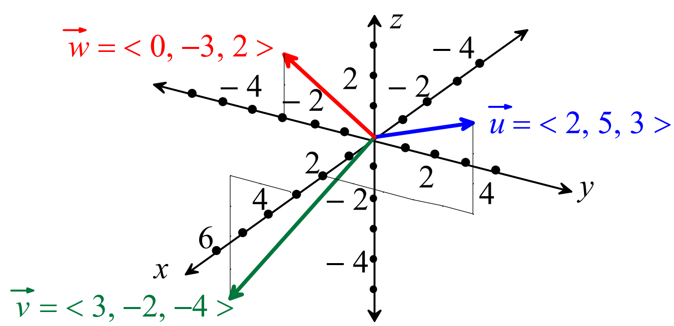
The zero vector is simply the function $z(x)$ which outputs the value 0 for all $a \in I$.

The negative of a function is simply defined by the function which outputs as its value of $-f(a)$, with input $x = a$.

How Can We Visualize Vectors?



Two Vectors, $\vec{u}$ and $\vec{v}$, in $\mathbb{R}^2$



Three Vectors, $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$

$\mathbb{R}^4$???

$\mathbb{P}^3$???

$F(\mathbb{R})$???

$m \times n$ *Matrices*

$$Mat(m,n) = \left\{ A \,\middle|\, A \text{ is an } m \times n \text{ matrix} \right\}$$

The Smallest Example

$$V = \left\{ \overrightarrow{0}_V \right\}$$

Addition? Scalar Multiplication?

We're Not in Kansas Anymore

$$\mathbb{R}^+ = \left\{ \vec{x} | x \in \mathbb{R}, \text{and} x > 0 \right\},$$

$$\vec{x} \oplus \vec{y} = \overrightarrow{xy} \qquad \text{(ordinary multiplication)}$$

$$r \odot \vec{x} = \overrightarrow{x^r} \qquad \text{(ordinary exponentiation)}$$

$$= \overrightarrow{e^{r \ln(x)}}$$

Identity element:

$$\vec{z} \oplus \vec{y} = \vec{y}$$

$$\vec{z} = ???$$

$$\overrightarrow{0}_{\mathbb{R}^+} = ???$$

Inverses:

$$\vec{x} \oplus \vec{y} = \overrightarrow{0}_{\mathbb{R}^+}$$

$$\vec{y} = ???$$

Last four Axioms:

$$(r + s) \odot \vec{x} = ???$$

$$r \odot (\vec{x} \oplus \vec{y}) = ???$$

$$(rs) \odot \vec{x} = ???$$

$$1 \odot \vec{x} = ???$$

Additional Properties of Vector Spaces

Theorem — The Uniqueness of the Zero Vector:

The ***zero vector*** $\vec{0}_V$ of any vector space $(V, \oplus, \odot)$ is ***unique***. This means that if $\vec{z} \in V$ is another vector that satisfies: $\vec{z} \oplus \vec{v} = \vec{v}$ for ***all*** $\vec{v} \in V$, then we must have: $\vec{z} = \vec{0}_V$.

Theorem — The Uniqueness of Additive Inverses:

The ***additive inverse*** $-\vec{v}$ of any vector $\vec{v} \in V$ in a vector space $(V, \oplus, \odot)$ is ***unique***. This means that if $\vec{n} \in V$ is another vector that satisfies: $\vec{v} \oplus \vec{n} = \vec{0}_V$, then we must have: $\vec{n} = -\vec{v}$.
As a further consequence: $-\vec{v} = -1 \odot \vec{v}$.

Theorem — The Multiplicative Properties of Zeroes:

Let $(V, \oplus, \odot)$ be a vector space, with zero vector $\vec{0}_V$. Then we have the following properties:

1. *The Multiplicative Property of the Scalar Zero:*

$$0 \odot \vec{v} = \vec{0}_V \text{ for all } \vec{v} \in V.$$

2. *The Multiplicative Property of the Zero Vector:*

$$r \odot \vec{0}_V = \vec{0}_V \text{ for all } r \in \mathbb{R}.$$

3. *The Zero-Factors Theorem:* For all $\vec{v} \in V$ and $r \in \mathbb{R}$:

$$r \odot \vec{v} = \vec{0}_V \textit{ if and only if } \text{either } r = 0 \textbf{\textit{ or }} \vec{v} = \vec{0}_V.$$

Definition — Axiom for Parallel Vectors:

Let $(V, \oplus, \odot)$ be a vector space, and let $\vec{u}, \vec{v} \in V$. We say that $\vec{u}$ and $\vec{v}$ are ***parallel to each other*** if there exists either $a \in \mathbb{R}$ ***or*** $b \in \mathbb{R}$ such that:

$$\vec{u} = a \odot \vec{v} \quad \textbf{\textit{or}} \quad \vec{v} = b \odot \vec{u}.$$

Consequently, this means that $\vec{\mathbf{0}}_V$ is parallel to ***all*** vectors $\vec{v} \in V$, since $\vec{\mathbf{0}}_V = 0 \odot \vec{v}$.

Things Don't Always Work Out

Example: Suppose $V = Mat(2,3)$, with vector addition defined as matrix addition, as before.

However, we will define scalar multiplication by:

$$r \odot A = r \odot \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

$$= \begin{bmatrix} ra_{1,1} & ra_{1,2} & ra_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

Do the Distributive Properties still hold?

Example: Suppose we let $V = \mathbb{R}^2$, but with addition defined by:

$$\langle x_1, y_1 \rangle \oplus \langle x_2, y_2 \rangle = \langle 2x_1 + 2x_2, y_1 + y_2 \rangle.$$

Scalar multiplication: same as before.

Is there a zero vector?

Does a vector have a negative?