3.3 Linearity Properties for Infinite Sets of Vectors

Definition: A non-empty set X is ***finite*** if the number of elements in the set is finite, that is, a positive integer n. In other words, we can choose to ***list*** the elements of X in some particular ***order***, say:

$$X = \{x_1, x_2, \ldots, x_n\},$$

where the list eventually ***terminates***. In this case, we call n the ***cardinality*** of our set, and we use the notation:

$$|X| = n,$$

pronounced as "the cardinality of X is n."

We agree that the ***empty set*** has cardinality 0, and we also consider it to be a finite set.

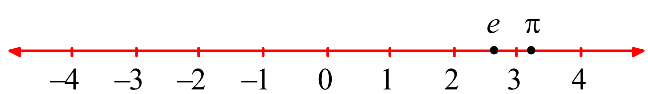
A set that is ***not*** finite is called an ***infinite set***.

Examples:

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

$$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

$$\mathbb{Q} = \left\{\frac{a}{b} \mid a \text{ and } b \text{ are integers, with } b \neq 0\right\},$$

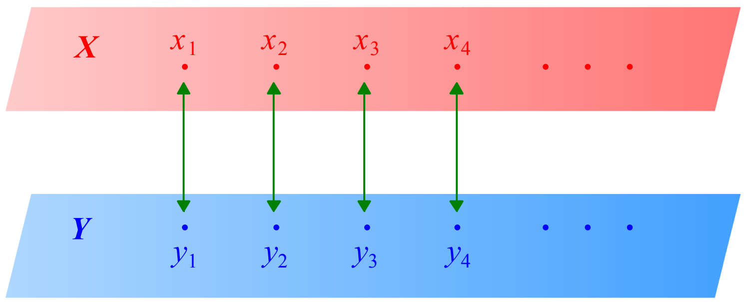


The Real Number Line With Some Members of $\mathbb{R}$

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Definition/Theorem — The Schroeder-Bernstein Theorem:

Suppose that X and Y are two sets (they can be finite or infinite). Then $|X| = |Y|$, that is, X and Y have the same cardinality, ***if and only if*** there exists a function $f : X \to Y$ which is both ***one-to-one*** and ***onto***.



A One-to-One Correspondence

Between Two Sets X and Y

Notation:

$$|\mathbb{N}| = \aleph_0.$$

Any set with cardinality $\aleph_0$ is called ***countable***.

Example:

$$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

$\mathbb{Q}$ is also countable; proof in the Exercises.

Example: $\mathbb{R}$ is ***not*** countable, in other words, it is ***impossible*** to list ***all*** the real numbers in a sequence.

Definitions — Comparing Cardinalities:

Suppose that X and Y are two sets (they can be finite or infinite). Then we say that $|X| < |Y|$, that is, the cardinality of X is ***strictly smaller*** than the cardinality of Y, if there exists a function $f : X \to Y$ which is ***one-to-one***, but there is ***no*** such function which is both one-to-one and ***onto***. In this case, we can also write: $|Y| > |X|$ and say that the cardinality of Y is ***strictly bigger*** than the cardinality of X.

We can also say that $|X| \leq |Y|$, that is, the cardinality of X is ***at most*** the cardinality of Y, if there exists a function $f : X \to Y$ which is ***one-to-one***. Such a function may or may not be onto. In this case, we can also write: $|Y| \geq |X|$ and say that the cardinality of Y is ***at least*** the cardinality of X.

$$|\mathbb{R}| = \aleph_1 > \aleph_0 = |\mathbb{N}|.$$

Any infinite set such as $\mathbb{R}$ whose cardinality is strictly bigger than $\aleph_0$ is called ***uncountable***.

Theorem — Countable and Uncountable Sets of Numbers:

The set of ***natural numbers***, ***integers***, and ***rational numbers*** are all ***countable:***

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0.$$

However, the set of ***real numbers***, the set of ***irrational numbers***, and all ***intervals*** of the real number line that contain at least two points are all ***uncountable*** and have cardinality $\aleph_1$:

$$|\mathbb{R}| = |\mathbb{R} - \mathbb{Q}| = |(a,b)| = |[a,b]| = |[a,b)| = |(a,b]| = |\mathbb{R}|$$

where $a < b \in \mathbb{R}$. More generally, these infinite intervals also have cardinality $\aleph_1$:

$$|(-\infty, b)| = |(-\infty, b]| = |(a, \infty)| = |[a, \infty)| = \aleph_1.$$

Describing Infinite Sets of Vectors

General set-builder notation:

$S = \{\vec{v}_i \mid i \in I\}$, where I is some indexing set (a subset of $\mathbb{R}$).

To avoid ambiguity, we will insist that $\vec{v}_i \neq \vec{v}_j$ if i and j are distinct indices in I. In other words, distinct indices correspond to distinct vectors, and vice versa.

Example:

$$S = \{1, x, x^2, x^3, \dots, x^n, \dots\},$$

Rewrite in set-builder notation.

Sets of ***even monomials*** and ***odd monomials:***

Example:

$$S_1 = \{e^{kx} \mid k \in \mathbb{Z}\} = \{\dots, e^{-3x}, e^{-2x}, e^{-x}, 1, e^{x}, e^{2x}, e^{3x}, \dots\}.$$

$$S_2 = \{e^{kx} \mid k \in \mathbb{Q}\}$$

$$S_3 = \{e^{kx} \mid k \in \mathbb{R}\}.$$

Linearity Concepts for Infinite Sets of Vectors

Suppose we are given the infinite set of vectors:

$$S = \{\vec{v}_i \,|\, i \in I\},$$

for some indexing set I. A ***finite subset*** of S can be listed explicitly, and written in roster form:

$$\{\vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_n}\},$$

where i_1, i_2, $\ldots$, i_n are numbers from I, which are called ***indices*** (the plural of ***index***), with $i_1 < i_2 < \cdots < i_n$. This notation is particularly important if I is not countable. This notation is called a ***double subscript notation***, because the subscripts of $\vec{v}$ also contain a subscript.

Definition: Let $(V, \oplus, \odot)$ be a vector space. Suppose that S is an infinite set of vectors from V. A ***linear combination*** of vectors from S can be constructed in the following way:

(a) Choose a finite subset of vectors: $\{\vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_n}\}$ from S.

(b) Choose a finite list of scalars $r_1,\ r_2,\ \ldots,\ r_n \in \mathbb{R}$, as before.

(c) Form the vector expression:

$$(r_1 \odot \vec{v}_{i_1}) \oplus (r_2 \odot \vec{v}_{i_2}) \oplus \cdots \oplus (r_n \odot \vec{v}_{i_n}).$$

Similarly, the Span of S, denoted $Span(S)$ as before, can be defined as the set of ***all possible linear combinations*** of vectors from finite subsets of S.

Based on the description above, we can construct $Span(S)$ as follows:

- Form ***all finite subsets*** of S : $\{\vec{v}_{i_1}\}$, $\{\vec{v}_{i_1}, \vec{v}_{i_2}\}$, $\{\vec{v}_{i_1}, \vec{v}_{i_2}, \vec{v}_{i_3}\}$, . . . and so on.

  In other words, form all subsets consisting of exactly one vector, exactly two vectors, exactly three vectors, and so on.
- For each of these subsets, form all possible linear combinations of these finite sets.
- Collect all of these linear combinations in one giant set which will be $Span(S)$.

Theorem: Suppose that $S = \{\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k, \ldots\}$ is a ***countable*** set of vectors from a vector space V. Then, a linear combination of the vectors in S is an expression of the form:

$$c_0\vec{v}_0 + c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k,$$

for some $k \in \mathbb{N}$ and coefficients c_0, c_1, …, c_k. Similarly, $Span(S)$ is the set of all linear combinations from S of the form given above.

Proof: A finite subset of n vectors from S has the form:

$$\{\vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_n}\},$$

where we can assume that $i_1 < i_2 < \cdots < i_n$, and these subscripts are all ***natural numbers***.

A linear combination of this finite set has the form:

$$r_1\vec{v}_{i_1} + r_2\vec{v}_{i_2} + \cdots + r_n\vec{v}_{i_n}.$$

Example:

$$\{\vec{v}_2, \vec{v}_5, \vec{v}_7\}$$

$$0 \cdot \vec{v}_0 + 0 \cdot \vec{v}_1 + c_2\vec{v}_2 + 0 \cdot \vec{v}_3 + 0 \cdot \vec{v}_4 + c_5\vec{v}_5 + 0 \cdot \vec{v}_6 + c_7\vec{v}_7$$

$$\{\vec{v}_0, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_7\}$$

Generalize:

$$c_0\vec{v}_0 + c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k,$$

Example: Let us consider the infinite set:

$$S = \{x^n \mid n \in \mathbb{N}\} = \{1, x, x^2, x^3, \ldots, x^n, \ldots\} \subset F(\mathbb{R}),$$

$$\begin{aligned} & c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot x^2 + c_3 \cdot x^3 + \cdots + c_n \cdot x^n \\ = \; & c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n. \end{aligned}$$

$$\begin{aligned} \mathbb{P} &= Span(\{1, x, x^2, x^3, \ldots, x^n, \ldots\}) \\ &= Span(\{x^n \mid n \in \mathbb{N}\}). \; \square \end{aligned}$$

Example: Consider the uncountable set:

$$S_3 = \{e^{kx} | k \in \mathbb{R}\} \subset F(\mathbb{R}).$$

To form a finite subset of n vectors, we pick n real numbers:

$$k_1 < k_2 < \cdots < k_n,$$

and form the set:

$$\{e^{k_1 x}, e^{k_2 x}, \ldots, e^{k_n x}\}.$$

A linear combination of this finite set therefore has the form:

$$c_1 e^{k_1 x} + c_2 e^{k_2 x} + \cdots + c_n e^{k_n x},$$

for some scalars $c_1, c_2, \ldots, c_n$.

Definition: Suppose that $S = \{\vec{v}_i \,|\, i \in I\}$ is an ***infinite*** set of vectors. We will say that S is ***linearly independent*** if ***every finite subset*** of S is linearly independent. This means that we must form ***every finite subset*** of S, in the form $\{\vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_n}\}$, where ever $i_j \in I$, and test whether or not this finite subset is independent. As soon as ***one*** finite subset is dependent, then S itself is dependent. However, if ***all*** finite subsets are independent, then S is independent.

Theorem: Suppose that $S = \{\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n, \dots\}$ is a ***countable*** set of vectors from a vector space V. Then, S is linearly independent ***if and only if*** every finite subset of the form:

$$\{\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n\}$$

is linearly independent, for ***every*** $n \in \mathbb{N}$. Similarly, if $S = \{\vec{v}_i \,|\, i \in I\}$, where $I \subseteq \mathbb{R}$, then S is linearly independent ***if and only if*** every finite subset of the form:

$$\{\vec{v}_{i_1}, \vec{v}_{i_2}, \dots, \vec{v}_{i_n}\}$$

is linearly independent, for all indices $i_1 < i_2 < \cdots < i_n$, where n is a positive integer.

Example: Let us return to $S = \{1, x, x^2, x^3, \dots, x^n, \dots\} \subset F(\mathbb{R})$.

Is S dependent or independent?

Example: Let us decide if the infinite uncountable set:

$$S_3 = \{e^{kx} | k \in \mathbb{R}\}$$

is linearly dependent or independent. We saw that every finite subset of S_3 has the form:

$$\{e^{k_1 x}, e^{k_2 x}, \dots, e^{k_n x}\},$$

where $k_1 < k_2 < \cdots < k_n$ and n is a positive integer.