3.4 Subspaces, Basis, and Dimension

Subspaces

Definition: A ***non-empty*** subset W of a vector space $(V, \oplus, \odot)$ is called a ***subspace*** of V if W is ***closed*** under vector addition and scalar multiplication.

In other words, for all $\vec{w}_1$ and $\vec{w}_2 \in W$, and $k \in \mathbb{R}$: $\vec{w}_1 \oplus \vec{w}_2 \in W$, and $k \odot \vec{w}_1 \in W$.

As before, we write $W \trianglelefteq V$, and we refer to V as the ***ambient space*** of W.

All the equations in Axioms 3, 4, 7, 8, 9 and 10 are automatically satisfied (we say they are ***inherited*** from V).

How about $\vec{0}_V$ and $-\vec{w}$?

Theorem: Let W be a non-empty subset of $(V, \oplus, \odot)$.

Then: W is a ***subspace*** of V ***if and only if*** $(W, \oplus, \odot)$ is ***itself*** a ***vector space***.

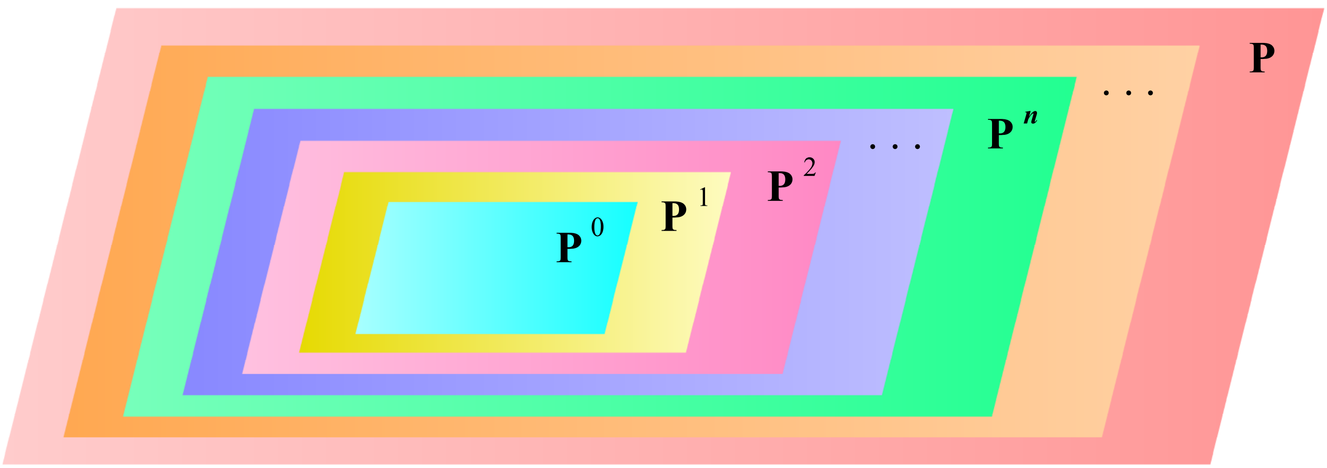
As before, the Span of a set of vectors is one of the easiest ways to construct a subspace of a vector space:

Theorem: Suppose $S = \{\vec{v}_i \,|\, i \in I\} \subset (V, \oplus, \odot)$, where $I \subset \mathbb{R}$ is some non-empty indexing set, and let $W = Span(S)$.

Then: $(W, \oplus, \odot)$ is a ***subspace*** of $(V, \oplus, \odot)$.

Subspaces of Function and Polynomial Spaces

$$\mathbb{P}^0 \trianglelefteq \mathbb{P}^1 \trianglelefteq \mathbb{P}^2 \trianglelefteq \mathbb{P}^3 \trianglelefteq \cdots \trianglelefteq \mathbb{P}^n \trianglelefteq \mathbb{P}^{n+1} \trianglelefteq \cdots \trianglelefteq \mathbb{P}$$



The Nesting of Polynomial Spaces

Subspaces of $F(I)$

The set $C(I)$ of ***continuous*** functions on I is a subspace of $F(I)$.

(sum also continuous? constant multiple?)

Now we focus on ***differentiable*** functions.

The sum of two differentiable functions and a scalar multiple of a differentiable function are again differentiable.

However, we will further require that the derivative of these functions also be continuous (there are indeed examples of differentiable functions whose derivatives are discontinuous.).

We call these functions $C^1(I)$, to denote that the first derivative is continuous.

But recall that a function which is differentiable on an open interval I is itself also continuous on I, and therefore the space $C^1(I)$ is a subspace of $C(I)$:

$$C^1(I) \trianglelefteq C(I)$$

Continuing with this logic, a function which is ***twice-differentiable*** on I with a ***continuous 2nd derivative*** also possesses a continuous first derivative, and so we can create:

$$C^2(I) \trianglelefteq C^1(I)$$

where $C^2(I)$ denotes all twice-differentiable functions whose second derivative is also continuous.

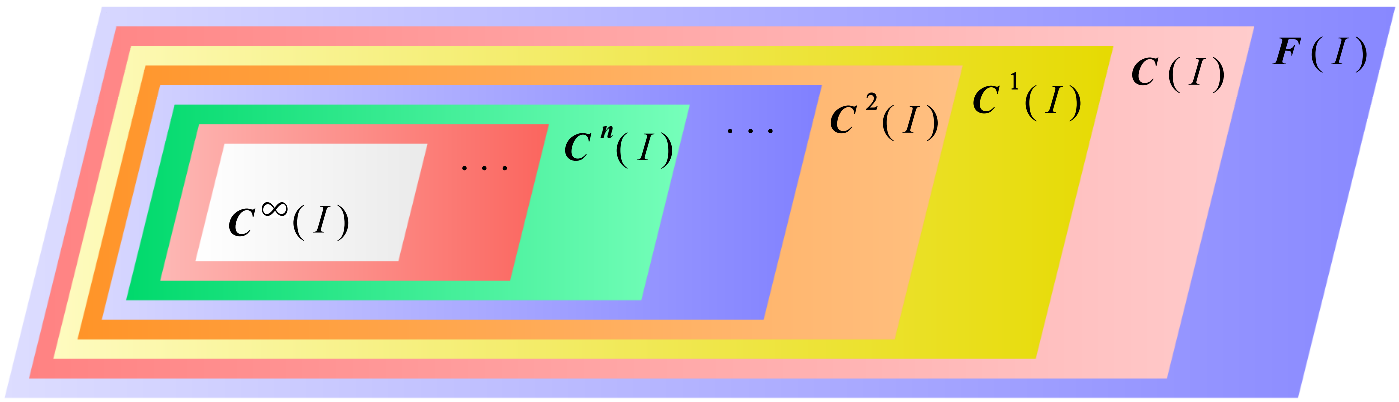
By Induction, we can define the subspace $C^n(I)$ consisting of all functions which are differentiable n times, and whose n^{th} derivative is also continuous.

$$F(I) \trianglerighteq C(I) \trianglerighteq C^1(I) \trianglerighteq C^2(I) \trianglerighteq \cdots$$
$$\trianglerighteq C^n(I) \trianglerighteq C^{n+1}(I) \trianglerighteq \cdots$$

Finally, we also have the subspace $C^\infty(I)$ of functions which have derivatives of all possible orders, and all of whose derivatives are also continuous.

We sometimes call this subspace the set of ***real analytic*** or ***smooth functions***.

Our friends the polynomials, the sine and cosine functions, and the exponential functions of any base, are all members of $C^\infty(I)$:



The Nesting of Spaces of Continuous and n – Differentiable Functions

Basis for a Vector Space

Definition — Basis for a Vector Space:

A set of vectors B from a vector space $(V, \oplus, \odot)$ is a ***basis*** for V if it is ***linearly independent*** and ***Spans*** V.

We will agree that the zero vector space $V = \left\{ \vec{\mathbf{0}}_V \right\}$ does not have a basis, since any set containing $\vec{\mathbf{0}}_V$ is automatically dependent.

Theorem — Uniqueness of Representation:

Suppose that $S = \{\vec{v}_i \mid i \in I\}$, for some non-empty indexing set I, is a set of vectors from some vector space $(V, \oplus, \odot)$.

Then: S is a basis for $(V, \oplus, \odot)$ ***if and only if*** every vector $\vec{v} \in V$ can be represented ***uniquely*** as a linear combination of a ***finite*** subset of vectors $\{\vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_k}\}$ from S:

$$\vec{v} = c_1\vec{v}_{i_1} + c_2\vec{v}_{i_2} + \cdots + c_k\vec{v}_{i_k}.$$

To construct a basis for a subspace, we will need:

Theorem — The Extension Theorem:

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a ***finite***, linearly ***independent*** set of vectors from some vector space $(V, \oplus, \odot)$, and suppose $\vec{v}_{n+1}$ is ***not*** a member of $Span(S)$.

Then, the ***extended*** set: $S^{/} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{v}_{n+1}\}$ is ***still linearly independent***.

Theorem — Existence of a Basis: Every non-zero vector space $(V, \oplus, \odot)$ has a ***basis*** B.

The Dimension of a Vector Space

Definition: A non-zero vector space $(V, \oplus, \odot)$ is called ***finite dimensional*** if we can find a ***finite*** set B which is a basis for V. We call such a set a ***finite basis*** for V.

Otherwise, we say that V is ***infinite dimensional***.

We will agree that the zero vector space $V = \left\{ \vec{\mathbf{0}}_V \right\}$ has dimension 0, and is also finite-dimensional.

To generalize the definition in Chapter 1, as before, we will need to generalize:

Theorem — The Dependent/Independent Sets from Spanning Sets Theorem:

Suppose we have a set of n vectors: $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\} \subset (V, \oplus, \odot)$,
and we form $W = Span(S)$.
Suppose now we randomly choose a set of m vectors from W to form a new set:

$$L = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}.$$

Then, we can conclude that: ***if*** $m > n$, ***then*** L is linearly ***dependent***.

Consequently, the ***contrapositive*** states that: ***if*** L is linearly ***independent***, ***then*** $m \leq n$.

Definition/Theorem — The Dimension of a Vector Space:

Any two ***bases*** for a ***finite-dimensional*** vector space $(V, \oplus, \odot)$ have exactly the same number of elements. We call this common number the ***dimension*** of V and is denoted as $dim(V)$. If $dim(V) = k$, we also say that V is a ***k-dimensional*** vector space.

Theorem: Let $(W, \oplus, \odot)$ be a subspace of a ***finite-dimensional*** vector space $(V, \oplus, \odot)$.

If $dim(V) = n$, then $dim(W) \leq n$, that is, $dim(W) \leq dim(V)$.

Furthermore, $dim(W) = n = dim(V)$ ***if and only if*** $W = V$.