5.1 Permutations and The Determinant Concept

Definition: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. The ***determinant*** of A is defined by:

$$det(A) = ad - bc.$$

Other common notations for $det(A)$ are $|A|$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Theorem: A 2×2 matrix A is ***invertible*** ***if and only if*** $det(A) \neq 0$.

Motivating a General Definition

If $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$, then:

$$det(A) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}.$$

Creating the 3×3 Determinant

Every term will now contain ***three factors***, of the form:

$$\pm\, a_{1,\_}\, a_{2,\_}\, a_{3,\_}$$

There will be six ways to fill in the blanks.

When will the term be positive, when will the term be negative?

Table of Terms

Term	Columns	Inversions	Number of Inversions
$\pm a_{1,1}a_{2,2}a_{3,3}$	$1,2,3$	none	0
$\pm a_{1,1}a_{2,3}a_{3,2}$	$1,3,2$	$3 > 2$	1
$\pm a_{1,2}a_{2,1}a_{3,3}$	$2,1,3$	$2 > 1$	1
$\pm a_{1,2}a_{2,3}a_{3,1}$	$2,3,1$	$2 > 1$ $3 > 1$	2
$\pm a_{1,3}a_{2,1}a_{3,2}$	$3,1,2$	$3 > 1$ $3 > 2$	2
$a_{1,3}a_{2,2}a_{3,1}$	$3,2,1$	$3 > 2$ $3 > 1$ $2 > 1$	3

Term	Number of Inversions	Coefficient	Final Term
$\pm a_{1,1}a_{2,2}a_{3,3}$	0	$+$	$+a_{1,1}a_{2,2}a_{3,3}$
$\pm a_{1,1}a_{2,3}a_{3,2}$	1	$-$	$-a_{1,1}a_{2,3}a_{3,2}$
$\pm a_{1,2}a_{2,1}a_{3,3}$	1	$-$	$-a_{1,2}a_{2,1}a_{3,3}$
$\pm a_{1,2}a_{2,3}a_{3,1}$	2	$+$	$+a_{1,2}a_{2,3}a_{3,1}$
$\pm a_{1,3}a_{2,1}a_{3,2}$	2	$+$	$+a_{1,3}a_{2,1}a_{3,2}$
$\pm a_{1,3}a_{2,2}a_{3,1}$	3	$-$	$-a_{1,3}a_{2,2}a_{3,1}$

Visualizing The Six Terms

$$\begin{bmatrix} \boxed{a_{1,1}} & a_{1,2} & a_{1,3} \\ a_{2,1} & \boxed{a_{2,2}} & a_{2,3} \\ a_{3,1} & a_{3,2} & \boxed{a_{3,3}} \end{bmatrix} \qquad \begin{bmatrix} \boxed{a_{1,1}} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & \boxed{a_{2,3}} \\ a_{3,1} & \boxed{a_{3,2}} & a_{3,3} \end{bmatrix}$$

$$+a_{1,1}a_{2,2}a_{3,3} \qquad -a_{1,1}a_{2,3}a_{3,2}$$

$$\begin{bmatrix} a_{1,1} & \boxed{a_{1,2}} & a_{1,3} \\ a_{2,1} & a_{2,2} & \boxed{a_{2,3}} \\ \boxed{a_{3,1}} & a_{3,2} & a_{3,3} \end{bmatrix} \qquad \begin{bmatrix} a_{1,1} & \boxed{a_{1,2}} & a_{1,3} \\ \boxed{a_{2,1}} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & \boxed{a_{3,3}} \end{bmatrix}$$

$$+a_{1,2}a_{2,3}a_{3,1} \qquad -a_{1,2}a_{2,1}a_{3,3}$$

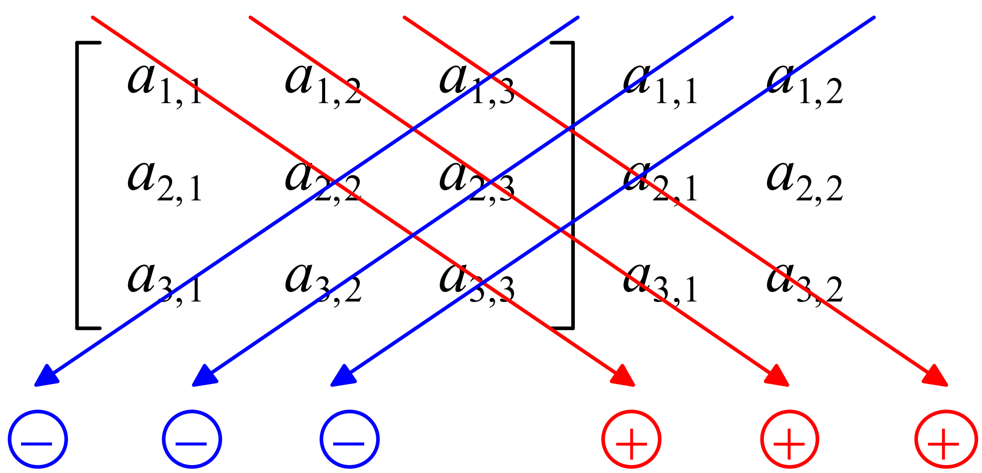
$$\begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} \\ \boxed{a_{2,1}} & a_{2,2} & a_{2,3} \\ a_{3,1} & \boxed{a_{3,2}} & a_{3,3} \end{bmatrix} \qquad \begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} \\ a_{2,1} & \boxed{a_{2,2}} & a_{2,3} \\ \boxed{a_{3,1}} & a_{3,2} & a_{3,3} \end{bmatrix}$$

$$+a_{1,3}a_{2,1}a_{3,2} \qquad -a_{1,3}a_{2,2}a_{3,1}$$

The 3×3 *Determinant*

Definition: If $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$, then:

$$det(A) = a_{1,1}a_{2,2}a_{33} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2}$$
$$- a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}$$



Permutation Theory

Definitions: A ***permutation*** of the set of integers $\{1, 2, \ldots, n\}$, is an ordered list consisting of these numbers, with each number appearing ***exactly once***. In other words, a permutation is a ***rearrangement*** of these numbers. We will label permutations with lowercase Greek letters such as σ or τ, and write them as:

$$\sigma = (i_1, i_2, \ldots, i_n).$$

We call i_k the k$^{\text{th}}$ ***component*** of σ.

Theorem: The number of permutations of $\{1, 2, \ldots, n\}$ is:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

Inversions

Definition: An ***inversion*** occurs in a permutation σ every time a component on the left is ***bigger*** than a component to its right.

We say that σ is ***even*** if there are an even number of inversions in σ, and σ is ***odd*** if there are an odd number of inversions in σ.

We define the ***sign*** of σ, denoted $\text{sgn}(\sigma)$, to be $+1$ if σ is ***even***, and -1 if σ is ***odd***.

Permutations as Bijections

A permutation σ can also be regarded as a ***bijection*** of the set $\{1, 2, \ldots, n\}$. Recall this means that σ is a one-to-one and onto function.

We can define the value of $\sigma(i)$ as the i^{th} component of σ.

Since σ is a bijection, it has an ***inverse***, denoted σ^{-1},
and if $\sigma(x) = y$, then $\sigma^{-1}(y) = x$.

The Graph of a Permutation

List out 1 through n in two horizontal rows, and draw an arrow from i to the number in the i^{th} component. Put a big space between the two rows to give you some wiggle room.

1 2 3 4 5 6 7 8

1 2 3 4 5 6 7 8

Counting Inversions from the Graph

Theorem: Suppose we represent σ, a permutation of $S = \{1, 2, \dots, n\}$, as a directed graph in the convention shown above. If $a < b \in S$, $\sigma(a) = c$, and $\sigma(b) = d$, then c is an ***inversion*** with respect to d (that is, $c > d$) ***if and only if*** the edge $a \to c$ ***intersects*** the edge $b \to d$ between our two lines of numbers.

The Effect of a Switch

Theorem: Let σ be a permutation of $\{1, 2, 3, \ldots, n\}$, and let σ' be the permutation obtained from σ by exchanging ***any*** two components of σ. Then:

$$sgn(\sigma') = -sgn(\sigma).$$

Ideas behind the proof:

$$\left(8, 5, 3, \boxed{2}, \boxed{7}, 1, 4, 6\right)$$

$$\left(8, \boxed{5}, 3, 2, 7, \boxed{1}, 4, 6\right)$$

The Balance of Even and Odd Permutations

Theorem: Exactly ***half*** of the $n!$ permutations of $\{1, 2, 3, .., n\}$ are ***even***, and ***half*** are ***odd***.

Even Permutations	Odd Permutations
$(1, 2, 3)$	$(2, 1, 3)$
$(2, 3, 1)$	$(3, 2, 1)$
$(3, 1, 2)$	$(1, 3, 2)$