5.2 A General Determinant Formula

Definition: Let A be an $n \times n$ matrix with entry $a_{i,j}$ in row i, column j, as usual. Then:

$$det(A) = \sum_{\substack{\text{all permutations} \\ \sigma \text{ of } \{1, 2, \ldots, n\}}} sgn(\sigma) \cdot a_{1,\sigma(1)} \cdot a_{2,\sigma(2)} \cdot \cdots \cdot a_{n-1,\sigma(n-1)} \cdot a_{n,\sigma(n)}$$

Example:

$$\pm\, a_{1,3}a_{2,5}a_{3,4}a_{4,1}a_{5,2}.$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \boxed{a_{2,5}} \\ a_{3,1} & a_{3,2} & a_{3,3} & \boxed{a_{3,4}} & a_{3,5} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \\ a_{5,1} & \boxed{a_{5,2}} & a_{5,3} & a_{5,4} & a_{5,5} \end{bmatrix}.$$

More generally, every term contains ***exactly one factor*** from each ***row***, and from each ***column***.

Basic Properties of $det(A)$

Theorem: Let A be an $n \times n$ matrix with a row of zeroes. Then $det(A) = 0$.

An Approach by Columns

Example:

$$-a_{1,3}a_{2,5}a_{3,4}a_{4,1}a_{5,2}.$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \boxed{a_{2,5}} \\ a_{3,1} & a_{3,2} & a_{3,3} & \boxed{a_{3,4}} & a_{3,5} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \\ a_{5,1} & \boxed{a_{5,2}} & a_{5,3} & a_{5,4} & a_{5,5} \end{bmatrix}.$$

Let us rearrange the factors so that the ***columns*** are in ***ascending order***:

Alternative Formula:

$$det(A) = \sum_{\substack{\text{all permutations} \\ \sigma \text{ of } \{1,2,\ldots,n\}}} sgn(\sigma) \cdot a_{\sigma(1),1} \cdot a_{\sigma(2),2} \cdot \cdots \cdot a_{\sigma(n-1),n-1} \cdot a_{\sigma(n),n}$$

The Determinant of the Transpose Matrix

Theorem: Let A be an $n \times n$ matrix. Then $det(A) = det(A^\top)$.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} \\ a_{2,1} & \boxed{a_{2,2}} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & \boxed{a_{3,4}} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

$$A^\top = \begin{bmatrix} a_{1,1} & a_{2,1} & a_{3,1} & \boxed{a_{4,1}} \\ a_{1,2} & \boxed{a_{2,2}} & a_{3,2} & a_{4,2} \\ \boxed{a_{1,3}} & a_{2,3} & a_{3,3} & a_{4,3} \\ a_{1,4} & a_{2,4} & \boxed{a_{3,4}} & a_{4,4} \end{bmatrix}.$$

Replace the Word ROW with COLUMN

Theorem: Let A be an $n \times n$ matrix with a ***column*** of zeroes. Then $det(A) = 0$.

Matrices with Proportional Rows or Columns

Theorem: Let A be an $n \times n$ matrix with two ***proportional*** rows (or, in particular, two ***equal*** rows). Then $det(A) = 0$. Similarly, a matrix with proportional columns also has zero determinant.

Typical term:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} \\ ka_{1,1} & \boxed{ka_{1,2}} & ka_{1,3} & ka_{1,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & \boxed{a_{3,4}} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

What other term will *cancel* this term?

Determinants of Triangular Matrices

Theorem: Let A be an upper or a lower triangular matrix, that is, $a_{i,j} = 0$ for all $i > j$, or $a_{i,j} = 0$ for all $i < j$. Then:

$$det(A) = a_{1,1} \bullet a_{2,2} \bullet \cdots \bullet a_{n-1,n-1} \bullet a_{n,n},$$

that is, the product of the ***diagonal*** entries. In particular:

$$\text{if } D = Diag(d_1, d_2, \ldots, d_n), \text{ then}$$

$$det(D) = d_1 \bullet d_2 \bullet \cdots \bullet d_n.$$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & a_{2,2} & a_{2,3} & a_{2,4} \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & 0 & a_{4,4} \end{bmatrix}.$$

Example:

$$A = \begin{bmatrix} 7 & 12 & 753 & 2^{12} \\ 0 & 4 & \sqrt{\pi} & 0 \\ 0 & 0 & -2 & 1/e \\ 0 & 0 & 0 & 1/14 \end{bmatrix}.$$

Determinants of Elementary Matrices

Theorem: Suppose E is an ***elementary*** matrix. If E is obtained from I_n by:

1. multiplying row i by $k \neq 0$, then:

$$det(E) = k.$$

2. exchanging row i and row j, then:

$$det(E) = -1.$$

3. adding k times row i to row j, then:

$$det(E) = 1.$$

Consequently, the determinant of every elementary matrix is ***non-zero***.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

The Effect of Row Operations

Theorem: Let A be an $n \times n$ matrix. Suppose B is obtained from A by:

1. ***multiplying*** row i of A by $k \neq 0$. Then:

$$det(B) = k \cdot det(A).$$

2. ***exchanging*** row i and row j of A. Then:

$$det(B) = -det(A).$$

3. ***adding*** k times row i of A to row j of A. Then:

$$det(B) = det(A).$$

Analogous statements can be made by replacing the word ***row*** with the word ***column***.

Consequently if E is the elementary matrix corresponding to the row operation performed on A to produce B, then $B = E \cdot A$, and so:

$$det(B) = det(E \cdot A) = det(E) \cdot det(A).$$

In particular:

$$det(k \cdot A) = k^n \cdot det(A).$$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} \\ a_{2,1} & \boxed{a_{2,2}} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & \boxed{a_{3,4}} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} \\ a_{2,1} & \boxed{a_{2,2}} & a_{2,3} & a_{2,4} \\ k \cdot a_{3,1} & k \cdot a_{3,2} & k \cdot a_{3,3} & \boxed{k \cdot a_{3,4}} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & \boxed{a_{3,4}} \\ a_{2,1} & \boxed{a_{2,2}} & a_{2,3} & a_{2,4} \end{bmatrix}$$

For a Type 3 matrix we will need:

Lemma: Let A, B and C be $n \times n$ matrices that have all entries equal except for the entries in row i. Suppose row i of C is the sum of row i of A and row i of B. Then:

$$det(C) = det(A) + det(B).$$

Warning: This Theorem is not saying that $C = A + B$,

nor is it saying that $det(A + B) = det(A) + det(B)$.

In fact, in general, this equation is ***false***:

most of the time, $det(A + B) \neq det(A) + det(B)$.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} \\ a_{2,1} & \boxed{a_{2,2}} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & \boxed{a_{3,4}} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

$$B = \begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} \\ a_{2,1} & \boxed{a_{2,2}} & a_{2,3} & a_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & \boxed{b_{3,4}} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

$$C = \begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} \\ a_{2,1} & \boxed{a_{2,2}} & a_{2,3} & a_{2,4} \\ a_{3,1} + b_{3,1} & a_{3,2} + b_{3,2} & a_{3,3} + b_{3,3} & \boxed{a_{3,4} + b_{3,4}} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

Back to Type 3 operations. Apply Lemma to B_3.

$$B_3 = \begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} \\ a_{2,1} & \boxed{a_{2,2}} & a_{2,3} & a_{2,4} \\ a_{3,1} + ka_{1,1} & a_{3,2} + ka_{1,2} & a_{3,3} + ka_{1,3} & \boxed{a_{3,4} + ka_{1,4}} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} \\ a_{2,1} & \boxed{a_{2,2}} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & \boxed{a_{3,4}} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

$$B = \begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} \\ a_{2,1} & \boxed{a_{2,2}} & a_{2,3} & a_{2,4} \\ k \cdot a_{1,1} & k \cdot a_{1,2} & k \cdot a_{1,3} & \boxed{k \cdot a_{1,4}} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

Finding det(A) Using Row and Column Operations

Idea: Perform row operations (like Gauss-Jordan) until we get an upper or lower triangular matrix. Account for all Type 1 and 2 operations you perform.

Example: Let us compute the determinant of:

$$A = \begin{bmatrix} 8 & -2 & 3 & -7 \\ -3 & 0 & 4 & 8 \\ 6 & 2 & -1 & -5 \\ 5 & -9 & -2 & 9 \end{bmatrix}$$

Correct answer: -3410