5.3 Properties of Determinants and Cofactor Expansion

Theorem: Let A be an $n \times n$ matrix. Then A is ***invertible*** ***if and only if*** $det(A)$ is ***non-zero***.

Find the rref of A:

$$R = E_t \cdot \cdots \cdot E_2 \cdot E_1 \cdot A$$

$$\begin{aligned} & det(R) \\ = & det(E_t \cdot \cdots \cdot E_2 \cdot E_1 \cdot A) \\ = & det(E_t) \cdot \cdots \cdot det(E_2) \cdot det(E_1) \cdot det(A) \end{aligned}$$

What do we know about the determinant of any elementary matrix?

The Multiplicative Property of Determinants

Theorem: Let A and B be $n \times n$ matrices. Then:

$$det(A \cdot B) = det(A) \cdot det(B)$$

Case 1: A ***is invertible.***

$$A = E_t \cdot \cdots \cdot E_2 \cdot E_1$$

$$\begin{aligned} det(A) &= det(E_t \cdot \cdots \cdot E_2 \cdot E_1) \\ &= det(E_t) \cdot \cdots \cdot det(E_2) \cdot det(E_1) \end{aligned}$$

$$\begin{aligned} det(A \cdot B) &= det(E_t \cdot \cdots \cdot E_2 \cdot E_1 \cdot B) \\ &= det(E_t) \cdot \cdots \cdot det(E_2) \cdot det(E_1) \cdot det(B) \\ &= det(A) \cdot det(B) \end{aligned}$$

Case 2: A ***is not invertible.***

Then $det(A) = 0$.

Recall: $A \cdot B$ is invertible ***if and only if*** . . .

Can $A \cdot B$ be invertible?

Determinants of Powers

Theorem: Let A be any $n \times n$ matrix. Then for any positive integer k:

$$det(A^k) = det(A)^k.$$

Furthermore, if A is ***invertible***, then:

$$det(A^{-1}) = \frac{1}{det(A)} = det(A)^{-1}.$$

Thus, if A is invertible, then for any ***integer*** power k:

$$det(A^k) = det(A)^k.$$

Computing Determinants Using Cofactor Expansion

Look at 3 × 3 determinants:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

Group terms using the 1st row entries:

$$\begin{aligned} det(A) &= \boxed{a_{1,1}}a_{2,2}a_{33} + \boxed{a_{1,2}}a_{2,3}a_{3,1} + \boxed{a_{1,3}}a_{2,1}a_{32} \\ &\quad - \boxed{a_{1,3}}a_{2,2}a_{3,1} - \boxed{a_{1,1}}a_{2,3}a_{3,2} - \boxed{a_{1,2}}a_{2,1}a_{3,3} \\ &= a_{1,1}(a_{2,2}a_{33} - a_{2,3}a_{3,2}) + \\ &\quad a_{1,2}(a_{2,3}a_{3,1} - a_{2,1}a_{3,3}) + \\ &\quad a_{1,3}(a_{2,1}a_{32} - a_{2,2}a_{3,1}) \end{aligned}$$

Express the 1st and 3rd groups as ***determinants***:

$$a_{2,2}a_{33} - a_{2,3}a_{3,2} = \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix}$$

$$a_{2,1}a_{32} - a_{2,2}a_{3,1} = \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}.$$

The 2nd group needs an adjustment:

$$a_{1,2}(a_{2,3}a_{3,1} - a_{2,1}a_{3,3}) = -a_{1,2}(a_{2,1}a_{3,3} - a_{2,3}a_{3,1})$$

$$a_{2,1}a_{3,3} - a_{2,3}a_{3,1} = \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix}$$

Row 1 is not Special.
Try Column 2!

$$\begin{bmatrix} a_{1,1} & \boxed{a_{1,2}} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

Look for factors from Column 2:

$$\begin{aligned} det(A) &= a_{1,1}\boxed{a_{2,2}}a_{33} + \boxed{a_{1,2}}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}\boxed{a_{3,2}} \\ &\quad - a_{1,3}\boxed{a_{2,2}}a_{3,1} - a_{1,1}a_{2,3}\boxed{a_{3,2}} - \boxed{a_{1,2}}a_{2,1}a_{3,3} \\ &= \boxed{-}a_{1,2}(a_{2,1}a_{3,3} - a_{2,3}a_{3,1}) \\ &\quad \boxed{+}a_{2,2}(a_{1,1}a_{33} - a_{1,3}a_{3,1}) \\ &\quad \boxed{-}a_{3,2}(a_{1,1}a_{2,3} - a_{1,3}a_{2,1}). \end{aligned}$$

Associated 2×2 determinants:

$$\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix},$$

$$\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} \text{ and}$$

$$\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix}$$

How Do We Determine the Sign?

Coefficient:	Sign:
$a_{1,1}$	$+$
$a_{1,2}$	$-$
$a_{1,3}$	$+$
$a_{2,2}$	$+$
$a_{3,2}$	$-$

Minors and Cofactors

Definition: Let A be an $n \times n$ matrix. The determinant of the submatrix obtained from A by erasing its *ith* row and *jth* column is called the ***i,j-minor*** of A, denoted:

$$M_{i,j}(A).$$

The ***i,j-cofactor*** of A is the number:

$$C_{i,j}(A) = (-1)^{i+j} \cdot M_{i,j}(A).$$

+	−	+	−
−	+	−	+
+	−	+	−
−	+	−	+

Revisit the 3×3 Determinant:

Using Row 1:

$$\begin{aligned} det(A) &= a_{1,1}M_{1,1}(A) - a_{1,2}M_{1,2}(A) + a_{1,3}M_{1,3}(A) \\ &= a_{1,1}C_{1,1}(A) + a_{1,2}C_{1,2}(A) + a_{1,3}C_{1,3}(A) \end{aligned}$$

Using Column 2:

$$\begin{aligned} det(A) &= -a_{2,1}M_{2,1}(A) + a_{2,2}M_{2,2}(A) - a_{2,3}M_{2,3}(A) \\ &= a_{2,1}C_{2,1}(A) + a_{2,2}C_{2,2}(A) + a_{2,3}C_{2,3}(A) \end{aligned}$$

Cofactor Expansion

Theorem: Let A be an $n \times n$ matrix. We can compute the determinant of A
by a ***cofactor expansion along row i:***

$$det(A) = a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \cdots + a_{i,n}C_{i,n},$$

or a ***cofactor expansion along column j:***

$$det(A) = a_{1,j}C_{1,j} + a_{2,j}C_{2,j} + \cdots + a_{n,j}C_{n,j}.$$

The Best of Both Worlds

Recursively use a combination of two techniques:

- Perform row or column operations in order to produce a row or column with mostly 0's.
- Perform a cofactor expansion along this row or column.
- Apply these two techniques to the resulting cofactors.

Example: Re-compute the determinant of the last Example from Section 5.2 Lecture File.