6.2 The Geometry of Eigentheory and Computational Techniques

The Kernel as an Eigenspace

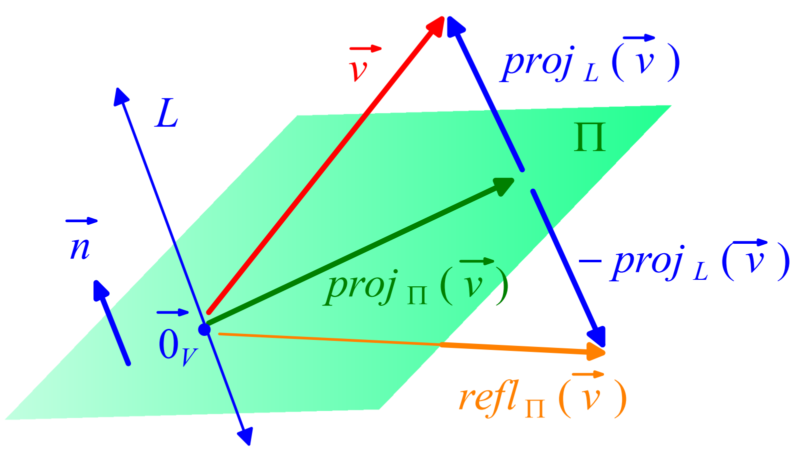
Theorem — Addenda to the Really Big Theorem on Invertibility:

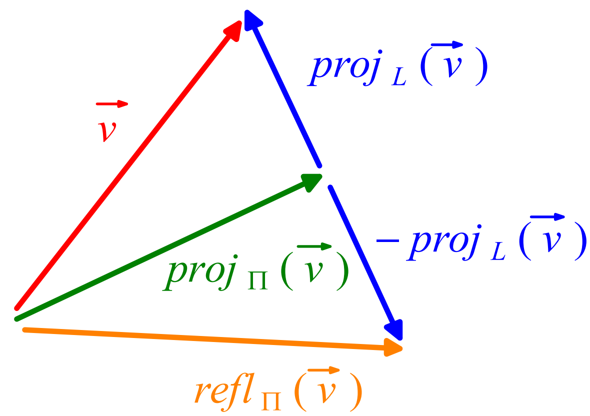
Let A be an $n \times n$ matrix. Then, the condition that A is ***invertible*** is equivalent to the following:

23. $det(A)$ is ***not*** 0.

24. $\lambda = 0$ is ***not*** an ***eigenvalue*** for A.

Projection and Reflection Operators





The Integer Roots Theorem:

Let $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$ be a polynomial with ***integer*** coefficients, and $c_0 \neq 0$. Then, all the rational roots of $p(x)$ are in fact integers, and if $x = c$ is an integer root of $p(x)$, then c is a ***factor*** of the constant coefficient c_0.

Example: What are the possible integer roots of:

$$p(\lambda) = \lambda^3 - 3\lambda^2 - 34\lambda + 120\ ?$$

The Rational Roots Theorem:

Let $q(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ be a polynomial with ***integer*** coefficients, with $c_0 \neq 0$. Then, all the rational roots of $q(x)$ are of the form $x = c/d$, where c is a factor of the constant coefficient c_0 and d is a factor of the leading coefficient c_n.

Example: What are the possible rational roots of:

$$p(\lambda) = \lambda^3 - \frac{7}{4}\lambda^2 - \frac{13}{36}\lambda + \frac{5}{6} \quad ?$$

Theorem (Descartes' Rule of Signs): The number of positive roots of a polynomial $p(x)$ with ***real*** coefficients is equal to the number of sign changes in consecutive coefficients of $p(x)$, or less than this number by an even amount. Similarly, the number of negative roots of $p(x)$ is the number of sign changes in consecutive coefficients of $p(-x)$, or less than this number by an even amount.

Theorem: Let $p(x)$ be a polynomial with ***odd degree***. Then $p(x)$ has at least one ***real*** root.

Example:

$$\begin{bmatrix} -25 & 11 & 11 \\ -132 & 63 & 66 \\ 66 & -33 & -36 \end{bmatrix}$$

Check: $p(\lambda) = \lambda^3 - 2\lambda^2 - 39\lambda - 72$