7.1 Inner Product Spaces

Definition (The Axioms of an Inner Product Space):
Let V be a vector space. An ***inner product*** on V is a ***bilinear form*** $\langle \mid \rangle$ on V, that is, a ***function*** that takes ***two vectors*** $\vec{u}$, $\vec{v} \in V$, and produces a ***scalar***, denoted $\langle \vec{u} | \vec{v} \rangle$, such that the following properties are satisfied by all vectors $\vec{u}$, $\vec{v}$ and $\vec{w} \in V$:

1. ***The Symmetric Property***

$$\langle \vec{u} | \vec{v} \rangle = \langle \vec{v} | \vec{u} \rangle$$

2. ***The Additive Property***

$$\langle \vec{u} + \vec{v} | \vec{w} \rangle = \langle \vec{u} | \vec{w} \rangle + \langle \vec{v} | \vec{w} \rangle$$

3. ***The Homogenous Property***

$$\langle k \cdot \vec{u} | \vec{v} \rangle = k \cdot \langle \vec{u} | \vec{v} \rangle$$

4. ***The Positive Property***

$$\text{If } \vec{v} \neq \vec{\mathbf{0}}_V, \text{ then } \langle \vec{v} | \vec{v} \rangle > 0.$$

We also say that V is an ***inner product space*** under the inner product $\langle \mid \rangle$.

How About the Zero Vector?

Theorem: Let V be an inner product space. Then, for any $\vec{v} \in V$:

$$\left\langle \vec{v} \middle| \vec{\mathbf{0}}_V \right\rangle = \left\langle \vec{\mathbf{0}}_V \middle| \vec{v} \right\rangle = 0.$$

In particular:

$$\left\langle \vec{\mathbf{0}}_V \middle| \vec{\mathbf{0}}_V \right\rangle = 0.$$

Other Easy Consequences

$$\langle \vec{u} \,|\, k \cdot \vec{v} \rangle = k \cdot \langle \vec{u} \,|\, \vec{v} \rangle, \quad \text{and}$$

$$\langle \vec{u} \,|\, \vec{v} + \vec{w} \rangle = \langle \vec{u} \,|\, \vec{v} \rangle + \langle \vec{u} \,|\, \vec{w} \rangle$$

If $k = -1$, we also have:

$$\langle \vec{u} - \vec{v} \,|\, \vec{w} \rangle = \langle \vec{u} + (-1 \cdot \vec{v}) \,|\, \vec{w} \rangle = \langle \vec{u} \,|\, \vec{w} \rangle + \langle -1 \cdot \vec{v} \,|\, \vec{w} \rangle$$

$$= \langle \vec{u} \,|\, \vec{w} \rangle - \langle \vec{v} \,|\, \vec{w} \rangle \quad \text{and}$$

$$\langle \vec{u} \,|\, \vec{v} - \vec{w} \rangle = \langle \vec{u} \,|\, \vec{v} \rangle - \langle \vec{u} \,|\, \vec{w} \rangle.$$

Weighted Dot Products

Let $\vec{u}$ and $\vec{v}$ be members of $\mathbb{R}^n$, and

let $\gamma_1, \gamma_2 \ldots \gamma_n$ be ***positive*** real numbers.

Define:

$$\langle \vec{u} | \vec{v} \rangle = \gamma_1 u_1 v_1 + \gamma_2 u_2 v_2 + \cdots + \gamma_n u_n v_n$$

Inner Products Generated by Isomorphisms

We can generalize the dot product in $\mathbb{R}^n$ further by considering any ***isomorphism***:

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

(that is, a one-to-one and onto operator) and define a new inner product on $\mathbb{R}^n$ by:

$$\langle \vec{u} | \vec{v} \rangle = T(\vec{u}) \circ T(\vec{v})$$

Polynomial Evaluations

Let $p(x)$ and $q(x)$ be members of $\mathbb{P}^n$, and

let $c_1,\ c_2, \ldots,\ c_n,\ c_{n+1}$ be ***any*** real numbers.

Define:

$$\langle p(x) \,|\, q(x) \rangle = p(c_1)q(c_1) + p(c_2)q(c_2) + \cdots + p(c_n)q(c_n) + p(c_{n+1})q(c_{n+1}).$$

Inner Products Induced by Integrals

Consider $C(I)$, the vector space of all ***continuous*** functions on $I = [a, b]$.

Define:

$$\langle f(x) \,|\, g(x) \rangle = \int_a^b f(x) \cdot g(x)\, dx$$

This appears in Math 55 when constructing *Fourier Series*.

A Non-Example

(Non-)Example: Let $\mathbb{R}^2$ be given the bilinear form:

$$\langle \vec{u} | \vec{v} \rangle = u_1 v_2 + u_2 v_1$$