7.3 Orthonormal Sets and The Gram-Schmidt Algorithm

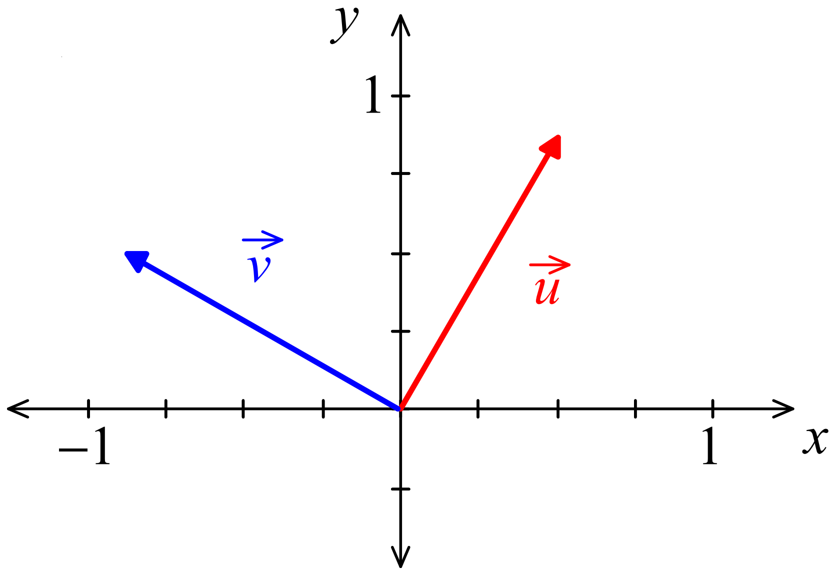
Definition: Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of vectors in an inner product space V. We say that S is an ***orthonormal set*** if:

$$\left\langle \vec{v}_i \,\middle|\, \vec{v}_j \right\rangle = 0 \text{ if } i \neq j, \text{ and}$$

$$\left\langle \vec{v}_i \,\middle|\, \vec{v}_i \right\rangle = 1 \text{ for } i = 1..k.$$

If we remove the condition that each member of S must be a unit vector but insist that all of the vectors be ***non-zero***, we call S an ***orthogonal set***.

The Orthonormal Sets in $\mathbb{R}^2$



$$\vec{v} = \left\langle -\sqrt{3}/2, 1/2 \right\rangle \text{ and } \vec{u} = \left\langle 1/2, \sqrt{3}/2 \right\rangle$$

More generally:

$$\{\langle \cos(\theta), \sin(\theta) \rangle, \langle -\sin(\theta), \cos(\theta) \rangle\} \text{ or}$$

$$\{\langle \cos(\theta), \sin(\theta) \rangle, \langle \sin(\theta), -\cos(\theta) \rangle\}$$

The Orthonormal Sets in Polynomial Spaces

Consider $\mathbb{P}^2$ under the inner product:

$$\left\langle p(x) \mid q(x) \right\rangle = p(-2)q(-2) + p(1)q(1) + p(3)q(3).$$

Challenge: Construct an orthonormal set which is as big as possible.

Hint: Think of strategically located zeroes.

Independence of Orthonormal/Orthogonal Sets

Theorem: An orthonormal set S in an inner product space V is ***linearly independent***.

Consequently, if $dim(V) = n$, and $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an orthonormal set, then $k \leq n$, and any set with more than n vectors cannot be orthonormal.

A similar Theorem with the word "orthogonal" replacing "orthonormal" is still true.

Orthonormal Bases

Definition/Theorem:

Let V be a finite dimensional inner product space with $\dim(V) = n$.

An orthonormal set $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ with n vectors is called an ***orthonormal basis*** for V.

If $\vec{v}$ is an arbitrary member of V, and S is an orthonormal basis for V, and:

$$\langle \vec{v} \rangle_S = \langle c_1, c_2, \ldots, c_n \rangle, \quad \text{then:}$$
$$c_i = \left\langle \vec{v} \mid \vec{u}_i \right\rangle.$$

In other words:

$$\vec{v} = \left\langle \vec{v} \mid \vec{u}_1 \right\rangle \cdot \vec{u}_1 + \left\langle \vec{v} \mid \vec{u}_2 \right\rangle \cdot \vec{u}_2 + \cdots + \left\langle \vec{v} \mid \vec{u}_n \right\rangle \cdot \vec{u}_n$$

The Gram-Schmidt Algorithm

Suppose that $\dim(V) = n$.

The input to the algorithm will be ***any*** basis $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ for V.

The output will be an ***orthogonal*** set:

$$S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\},$$

with the special property that:

$$Span(\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}) = Span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\})$$

for all $k = 1 \ldots n$.

By dividing each vector by its length, we can thus obtain an ***orthonormal*** basis for V.

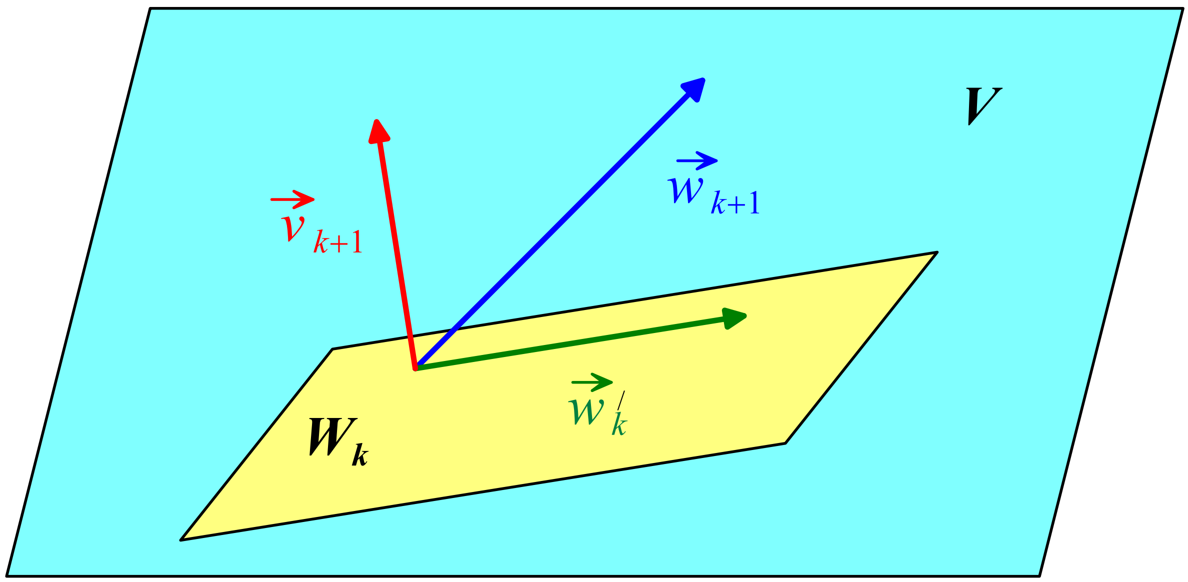
Theorem (The Gram-Schmidt Algorithm):

1. Start with any basis $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ for V.
2. Let $\vec{v}_1 = \vec{w}_1$. If $n = 1$, we are done,
   otherwise proceed to Step 3:
3. Let $\vec{v}_2 = \vec{w}_2 - \dfrac{\langle \vec{w}_2 \mid \vec{v}_1 \rangle}{\langle \vec{v}_1 \mid \vec{v}_1 \rangle}\vec{v}_1$.

   If $n = 2$, we are done,
   otherwise proceed to Step 4:
4. Let $\vec{v}_3 = \vec{w}_3 - \dfrac{\langle \vec{w}_3 \mid \vec{v}_1 \rangle}{\langle \vec{v}_1 \mid \vec{v}_1 \rangle}\vec{v}_1 - \dfrac{\langle \vec{w}_3 \mid \vec{v}_2 \rangle}{\langle \vec{v}_2 \mid \vec{v}_2 \rangle}\vec{v}_2$.

   If $n = 3$, we are done, otherwise for all $k \geq 3$,
   perform Step 5 until we have n orthogonal vectors:
5. Let

$$\vec{v}_{k+1} = \vec{w}_{k+1} - \frac{\langle \vec{w}_{k+1} \mid \vec{v}_1 \rangle}{\langle \vec{v}_1 \mid \vec{v}_1 \rangle}\vec{v}_1 - \frac{\langle \vec{w}_{k+1} \mid \vec{v}_2 \rangle}{\langle \vec{v}_2 \mid \vec{v}_2 \rangle}\vec{v}_2 - \cdots - \frac{\langle \vec{w}_{k+1} \mid \vec{v}_k \rangle}{\langle \vec{v}_k \mid \vec{v}_k \rangle}\vec{v}_k.$$



Constructing the Next Vector $\vec{v}_{k+1}$