1.2 The Span of a Set of Vectors

Definition: The ***Span*** of a non-empty set of vectors $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ from $\mathbb{R}^n$ is the set of ***all possible linear combinations*** of the vectors in the set. We write:

$$\begin{aligned} Span(S) &= Span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}) \\ &= \{\boxed{x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k} \mid \\ &\quad x_1, x_2, \ldots, x_k \in \mathbb{R}\}. \end{aligned}$$

We note that the individual vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are all members of $Span(S)$, where we let $x_i = 1$ and all the other coefficients 0 in order to produce $\vec{v}_i$. Similarly, the zero vector $\vec{0}_n$ is also a member of $Span(S)$, where we make all the coefficients x_i zero to produce $\vec{0}_n$.

Theorem: In any $\mathbb{R}^n$: $Span\left(\left\{\vec{0}_n\right\}\right) = \left\{\vec{0}_n\right\}$.

$x\vec{0} = \vec{0}$

Theorem: For all $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$:

$$Span\Big(\overbrace{\left\{\vec{0}_n, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\right\}}^{k+1}\Big)$$
$$= Span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\})^{k}.$$

Theorem: $\mathbb{R}^n = Span(\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\})$.

already proved!

B/c: $\langle x_1, x_2, \ldots, x_n \rangle = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$

The Span of One Vector in $\mathbb{R}^2$

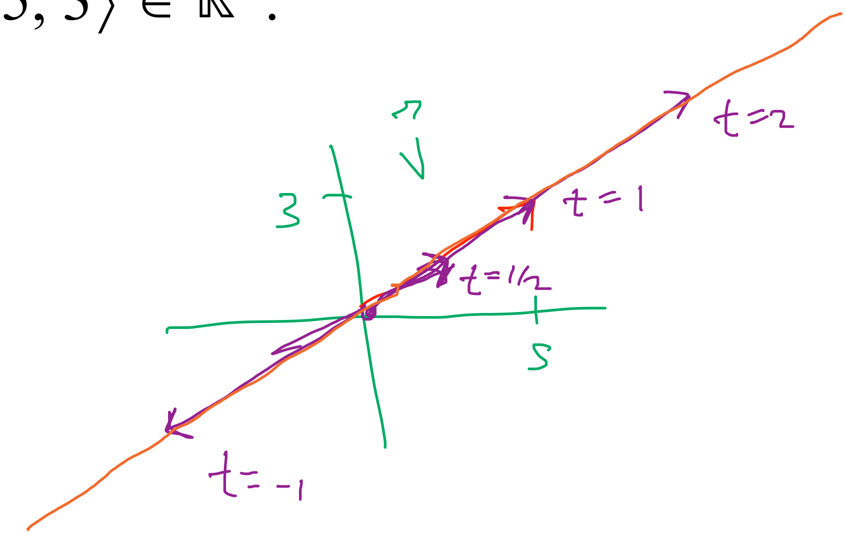
Example: Suppose that $\vec{v} = \langle 5, 3\rangle \in \mathbb{R}^2$.

Describe $Span(\{\vec{v}\})$.

$= \{ t\vec{v} \mid t \in \mathbb{R} \}$

get a line!

in plane

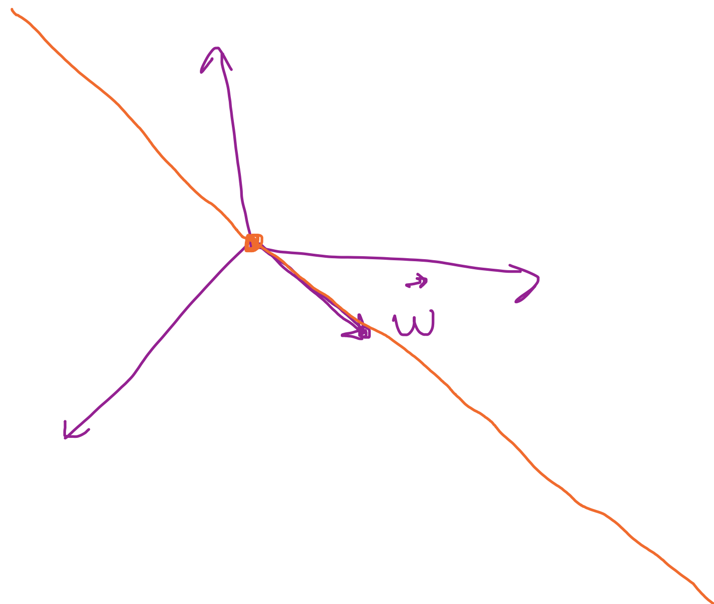


The Span of One Vector in $\mathbb{R}^3$

Example: Suppose that $\vec{w} = \langle -2, 1, -4 \rangle \in \mathbb{R}^3$. Describe $Span(\{\vec{v}\})$

$= \{ t\vec{v} \mid t \in \mathbb{R} \}$

get a Line in space!



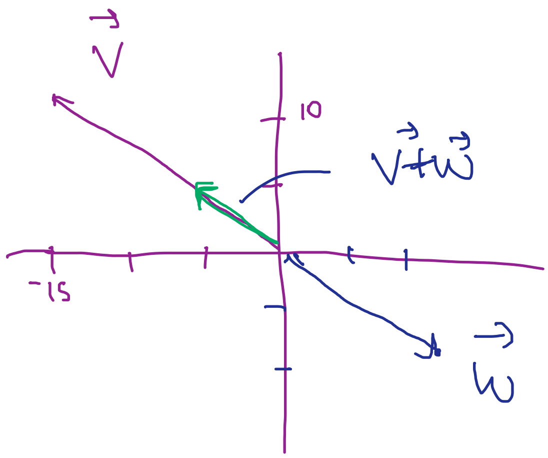
Lines in $\mathbb{R}^n$

Definition — Axiom for a Line:

If $\vec{v} \in \mathbb{R}^n$ is a ***non-zero*** vector, then $\boxed{Span(\{\vec{v}\})}$ is "geometrically" a ***line*** L in $\mathbb{R}^n$ passing through the origin.

The Span of Two Parallel Vectors

Example: Suppose that $\vec{v} = \langle -15, 10 \rangle$ and $\vec{w} = \langle 12, -8 \rangle \in \mathbb{R}^2$. Describe $Span(\{\vec{u}, \vec{v}\})$.



$Span(\{\vec{u}, \vec{v}\})$ = a Line!

$\vec{v} \parallel \vec{w}$?

Recall def. of $\parallel$:

$\exists\, k \in \mathbb{R}$ so that

$\vec{v} = k\vec{w}$ ✓

$\langle -15, 10 \rangle = k \langle 12, -8 \rangle$?

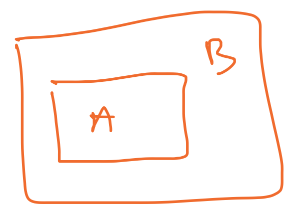
$-15 = k12 \rightarrow \boxed{k = -5/4}$ ✓

$10 = k(-8) \rightarrow k = \frac{10}{-8} = \frac{-5}{4}$ ✓

Theorem: If $\vec{u}$ and $\vec{v}$ are non-zero vectors in some $\mathbb{R}^n$ which are parallel to each other, then:

$$\underbrace{Span(\{\vec{u}, \vec{v}\})}_{A} = \underbrace{Span(\{\vec{v}\})}_{B} = Span(\{\vec{u}\}).$$

- $x_1\vec{u} + x_2\vec{v} \in span(\{\vec{u}, \vec{v}\})$
- $\vec{u} \parallel \vec{v}$ $\exists k \in \mathbb{R}$ so that $\vec{u} = k\vec{v}$



$$x_1\vec{u} + x_2\vec{v} = x_1(k\vec{v}) + x_2\vec{v}$$
$$= \underbrace{(x_1k + x_2)}_{\text{scaler}}\vec{v}$$

this says $x_1\vec{u} + x_2\vec{v} \in span(\{\vec{v}\})$.

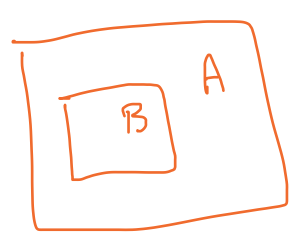
So $Span(\{\vec{u}, \vec{v}\}) \subseteq Span(\{\vec{v}\})$

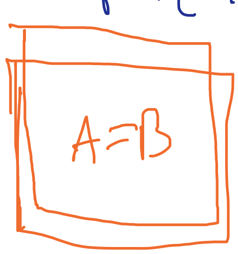
- WTS: $span(\{\vec{u}\}) \subseteq span(\{\vec{u}, \vec{v}\})$.

let $x_1\vec{u} \in span(\{\vec{u}\})$. let $x_2 = 0$ Then

$$x_1\vec{u} = x_1\vec{u} + 0\vec{v} = \underbrace{x_1\vec{u} + x_2\vec{v}} \in span(\{\vec{u}, \vec{v}\})$$

This proves $Span(\{\vec{u}\}) \subseteq span(\{\vec{u}, \vec{v}\})$.



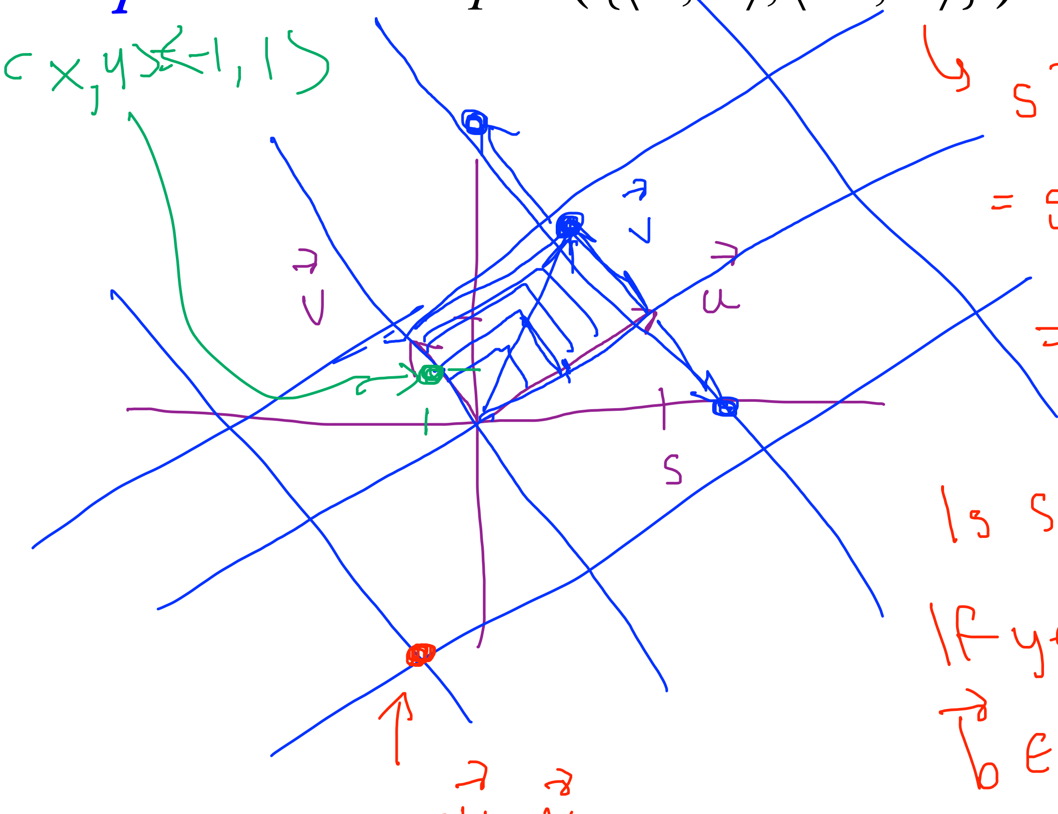


The Span of Two Non-Parallel Vectors in $\mathbb{R}^2$

Ans: it is all of $\mathbb{R}^2$!

$\vec{u}$ $\vec{v}$

Example: Describe *Span*$(\{\langle 5, 3\rangle, \langle -1, 2\rangle\})$ in $\mathbb{R}^2$.



$s\vec{u} + t\vec{v}$

$= s\langle 5,3\rangle + t\langle -1,2\rangle$

$= \langle 5s-t, 3s+2t\rangle$

Is Span $= \mathbb{R}^2$?

If yes, then any $\vec{b} \in \mathbb{R}^2$ is of the form

So, write $\vec{b} = \langle x,y\rangle$. If yes, then

$\langle x,y\rangle = \vec{b} = \langle 5s-t, 3s+2t\rangle$

(on board)

$\boxed{s = \frac{2x+y}{13}}$ $\boxed{t = 5\left[\frac{2x+y}{13}\right] - x}$

In general:

Theorem: If $\vec{u}, \vec{v} \in \mathbb{R}^2$ are ***non-parallel*** vectors, then:

$$\boxed{Span(\{\vec{u}, \vec{v}\}) = \mathbb{R}^2.}$$

yay!

In other words, ***any vector*** $\vec{w} \in \mathbb{R}^2$ can be expressed as a linear combination:

$$\vec{w} = r\vec{u} + s\vec{v},$$

yup!

for some scalars r and s.

if $\vec{w} \in \mathbb{R}^2$ then

$\vec{w} \in span(\{\vec{u}, \vec{v}\})$

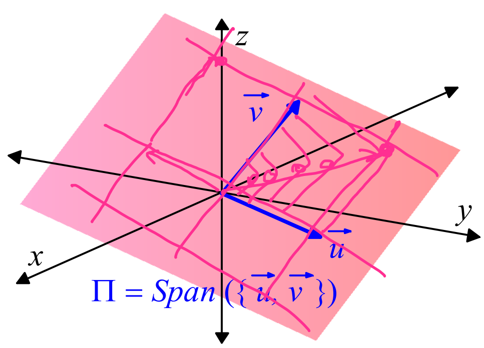
then

$\vec{w} = r\vec{u} + s\vec{v}$

The Span of Two Non-Parallel Vectors in $\mathbb{R}^3$

Definition — Axiom for a Plane in Cartesian Space:
If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$ that are ***not parallel*** to each other, then $Span(\{\vec{u}, \vec{v}\})$ is geometrically a ***plane*** Π in Cartesian space that passes through the origin (Π is the capital form of the lowercase Greek letter π).

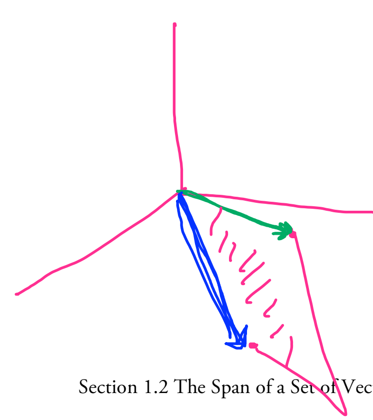
"P" "Π" in Greek.



$$Span(\{\vec{u}, \vec{v}\}) = \Pi$$

Example: $Span(\{\langle -2, 1, -3\rangle, \langle 5, 4, -3\rangle\})$.

plane through the origin.



The Cartesian Equation of a Plane

Definition: The ***Cartesian equation*** of a plane through the ***origin*** in Cartesian space, given in the form $\boxed{\Pi = Span(\{\vec{u}, \vec{v}\})}$, where $\vec{u}$ and $\vec{v}$ are not parallel, has the form:

$$ax + by + cz = 0,$$

$\langle a, b, c \rangle \cdot \langle x, y, z \rangle = 0$

$\vec{n} \cdot \vec{v} = 0$

for some constants, a, b and c, where at least one coefficient is non-zero.

Vector Eq Plane

$\Pi = \{ r\vec{u} + s\vec{v} \mid r, s \in \mathbb{R} \}$

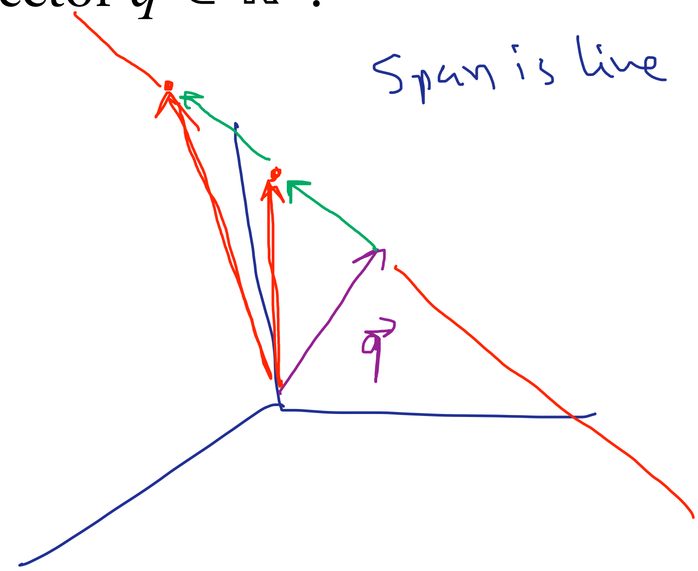
Translation of a Span

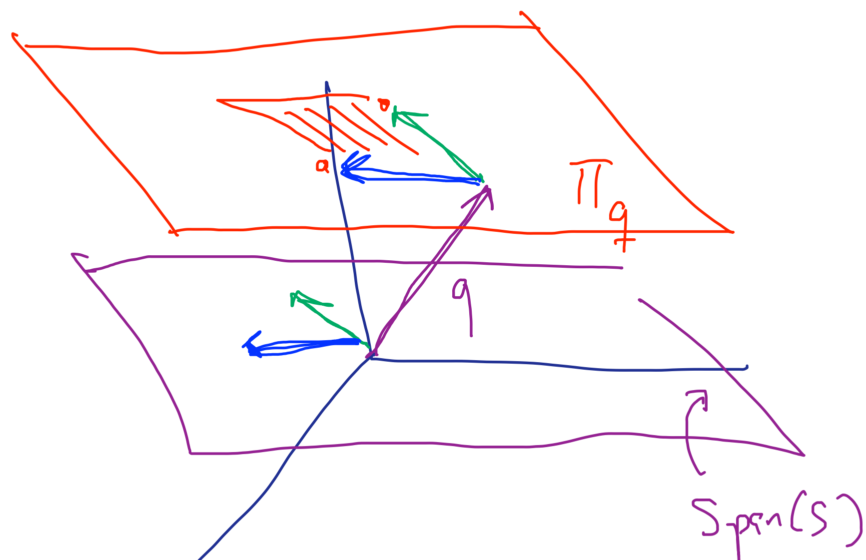
$$Q = \{\vec{q} + \vec{v} \mid \vec{v} \in Span(S)\},$$

$= \{\vec{q} + t\vec{v} \mid t \in \mathbb{R}\}$ $\vec{v}$ fixed some vector in span.

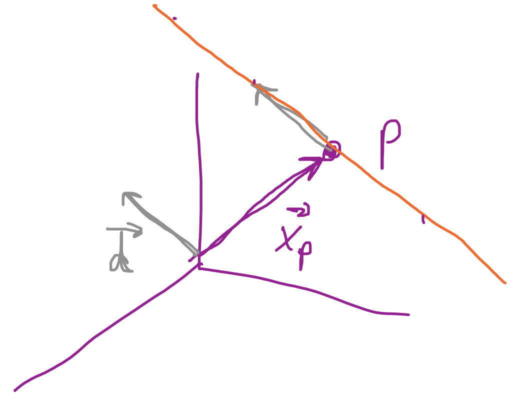
for some ***fixed*** non-zero vector $\vec{q} \in \mathbb{R}^n$.

Span is line





General Lines in $\mathbb{R}^n$



Definitions: A ***line*** L in $\mathbb{R}^n$ is the translate of the Span of a single ***non-zero*** vector $\vec{d} \in \mathbb{R}^n$:

$$\boxed{L = \left\{\vec{x}_p + t\vec{d} \,\middle|\, t \in \mathbb{R}\right\},}$$

EQ Line passing point P in the direction of $\vec{d}$

for some vector $\vec{x}_p \in \mathbb{R}^n$. We may think of $\vec{d}$ as a ***direction vector*** of L, and any non-zero multiple of $\vec{d}$ can also be used as a direction vector for L.

We see that by setting t to zero that $\vec{x}_p$ is a ***particular*** vector on the line L. We will also say that two ***distinct*** lines are ***parallel*** to each other if they are different translates of the same line through the origin.

General Lines in $\mathbb{R}^3$

Example: Consider the line L in Cartesian space passing through the point $(-5, 2, -3)$ and pointing in the direction of $\langle 2, 4, -7 \rangle$.

Definition: A line L in Cartesian space passing through the point (x_0, y_0, z_0), and with non-zero direction vector $\vec{d} = \langle a, b, c \rangle$ can be specified using a ***vector equation***, in the form:

$$\boxed{\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle}, \text{ where } t \in \mathbb{R}.$$

If ***none*** of the components of $\vec{d}$ are zero, we can obtain ***symmetric equations*** for L, of the form:

ie solve for t

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}.$$

Assume $a, b, c \neq 0$

$x = x_0 + ta \longrightarrow t = \frac{x - x_0}{a}$

$y = y_0 + tb \longrightarrow t = \frac{y - y_0}{b}$

$z = z_0 + tc \longrightarrow t = \frac{z - z_0}{c}$

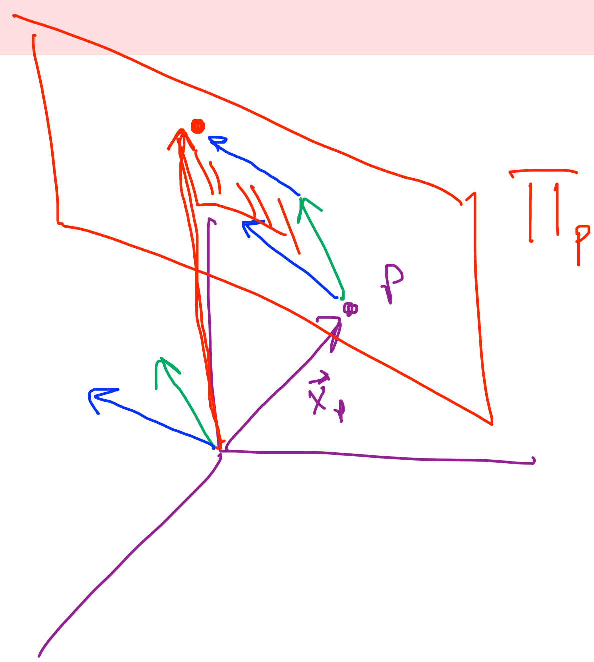
General Planes in $\mathbb{R}^n$

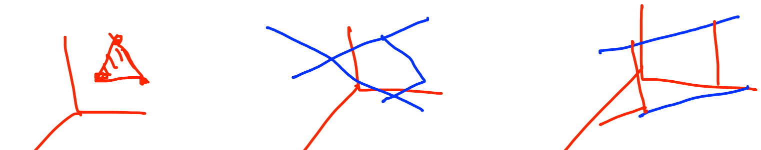
Definition: A ***plane*** Π in $\mathbb{R}^n$ is the translate of a Span of two ***non-parallel*** vectors $\vec{u}$ and $\vec{v} \in \mathbb{R}^n$:

$$\Pi_p = \{\vec{x} = \vec{x}_p + r\vec{u} + s\vec{v} \mid r, s \in \mathbb{R}\},$$

for some $\vec{x}_p \in \mathbb{R}^n$.

" parametric EQs of a plane "



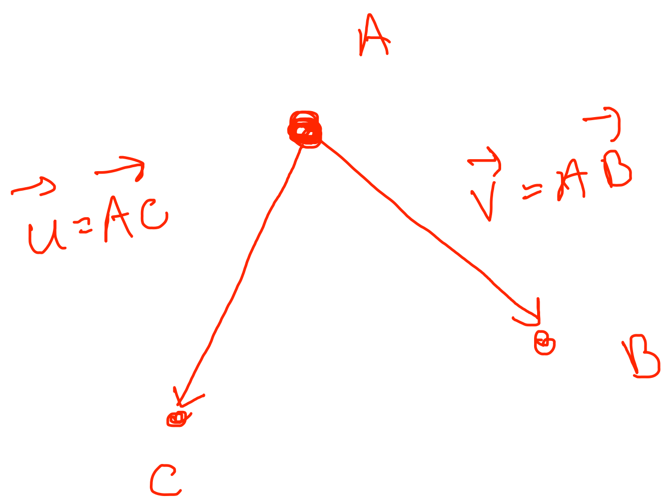


Some creative ways to specify a plane in Cartesian space:

- requiring the plane to contain three non-collinear points.
- requiring the plane to contain two intersecting lines.
- requiring the plane to contain two parallel lines.

Example: Find parametric equations and a Cartesian equation for the plane Π passing through $A(1,-3,2)$, $B(-1,-2,1)$ and $C(2,3,-1)$.

$$\Pi = \{ \vec{x}_A + s\vec{u} + t\vec{v} \mid s,t \in \mathbb{R} \}$$



Definition: A plane Π in Cartesian space can be specified using a ***Cartesian equation***, in the form:

$$ax + by + cz = d,$$

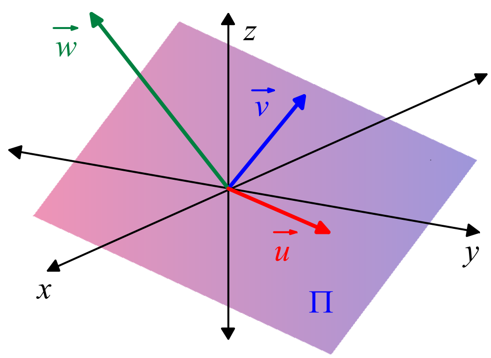
for some constants, a, b, c and d, where either a or b or c is non-zero. It is not unique, because we can multiply all the coefficients in the equation by the same non-zero constant k, and the resulting equation will again be a Cartesian equation for Π. The plane passes through the origin ***if and only if*** $d = 0$.

The Span of Three Non-Coplanar Vectors

Theorem: If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are ***non-coplanar*** vectors in $\mathbb{R}^3$, that is, none of these vectors is on the plane determined by the two others, then:

$$Span(\{\vec{u}, \vec{v}, \vec{w}\}) = \mathbb{R}^3.$$

In other words, any vector $\vec{z} \in \mathbb{R}^3$ can be expressed as a linear combination, $\vec{z} = r\vec{u} + s\vec{v} + t\vec{w}$, for some scalars r, s and t.



If $\vec{u}$, $\vec{v}$ and $\vec{w}$ Are *Non-Coplanar* Vectors in $\mathbb{R}^3$,

Then $Span(\{\vec{u}, \vec{v}, \vec{w}\}) = \mathbb{R}^3$