(not general vector space!)
special to $\mathbb{R}^n$

1.3 The Dot Product and Orthogonality

$u_i \in \mathbb{R}$

Definition: If $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ are vectors from $\mathbb{R}^n$, we define their ***dot product***:

$$\boxed{\vec{u} \circ \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.}$$

scalar or vector ?

Example: If $\vec{u} = \langle 4, -3, -6, 5, -2 \rangle$ and $\vec{v} = \langle 3, -5, 4, -7, -1 \rangle$, then:

$$\vec{u} \circ \vec{v} = (4)(3) + (-3)(-5) + (-6)(4) + (5)(-7) + (-2)(-1)$$

$$= 12 + 15 - 24 - 35 + 2$$

$$= \boxed{-30}$$

Length of a Vector

$\mathbb{R}^n$ · $\vec{v}+\vec{w} \in \mathbb{R}^n$
· $r\vec{v} \in \mathbb{R}^n$
· dot product new structure

Definitions: We define the ***length*** or ***norm*** or ***magnitude*** of a vector $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle \in \mathbb{R}^n$ as the non-negative number:

$$\boxed{\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.}$$

It follows directly from the definition of the dot product that:

$$\boxed{\|\vec{v}\|^2 = \vec{v} \circ \vec{v},} \text{ or in other words, } \|\vec{v}\| = \sqrt{\vec{v} \circ \vec{v}}.$$

A vector with length 1 is called a ***unit vector***. $\|\vec{v}\| = 1$

$$\vec{v} \circ \vec{v} = \langle v_1, v_2, \ldots, v_n \rangle \circ \langle v_1, v_2, \ldots, v_n \rangle$$

$$= v_1^2 + v_2^2 + \cdots + v_n^2$$

$$= \|\vec{v}\|^2$$

$\square$

Theorem: For any vector $\vec{v} \in \mathbb{R}^n$ and $k \in \mathbb{R}$: $\boxed{\|k\vec{v}\| = |k|\|\vec{v}\|}$.
In particular, if $\vec{v} \neq \vec{0}_n$, then $\boxed{\vec{u}_1 = \frac{1}{\|\vec{v}\|}\vec{v}}$ is the unit vector in the same direction as $\vec{v}$, and $\boxed{\vec{u}_2 = -\frac{1}{\|\vec{v}\|}\vec{v}}$ is the unit vector in the opposite direction as $\vec{v}$. Furthermore:

$$\boxed{\|\vec{v}\| = 0 \textit{ if and only if } \vec{v} = \vec{0}_n.}$$

$\in \mathbb{R}^5$

Example: The vector $\vec{v} = \langle 3, -2, 5, -4, -8 \rangle$ has length:

$$\|\vec{v}\| = \sqrt{9+4+25+16+64} = \sqrt{118}.$$

The two unit vectors parallel to $\vec{v}$ are:

$$\vec{u}_1 = \frac{1}{\sqrt{118}}\vec{v} = \left\langle \frac{3}{\sqrt{118}}, \frac{-2}{\sqrt{118}}, \frac{5}{\sqrt{118}}, \frac{-4}{\sqrt{118}}, \frac{-8}{\sqrt{118}} \right\rangle$$

= simplify ....

$$\vec{u}_2 = -\vec{u}_1 = \left\langle \frac{-3}{\sqrt{118}}, \frac{2}{\sqrt{118}}, \frac{-5}{\sqrt{118}}, \frac{4}{\sqrt{118}}, \frac{8}{\sqrt{118}} \right\rangle.$$

Properties of the Dot Product

MEMORIZE THESE

Theorem — Properties of the Dot Product:

For any vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and scalar $k \in \mathbb{R}$, we have:

1. *The Commutative Property*

$$\boxed{\vec{u} \circ \vec{v} = \vec{v} \circ \vec{u}.}$$

2. *The Right Distributive Property*

$$\boxed{\vec{u} \circ (\vec{v} + \vec{w}) = \vec{u} \circ \vec{v} + \vec{u} \circ \vec{w}.}$$

3. *The Left Distributive Property*

$$\boxed{(\vec{u} + \vec{v}) \circ \vec{w} = \vec{u} \circ \vec{w} + \vec{v} \circ \vec{w}.}$$

4. *The Homogeneity Property*

$$\boxed{(k \bullet \vec{u}) \circ \vec{v} = k(\vec{u} \circ \vec{v})} = \vec{u} \circ (k \bullet \vec{v}).$$

5. *The Zero-Vector Property*

$$\boxed{\vec{u} \circ \vec{0}_n = 0.}$$

6. *The Positivity Property*

$$\boxed{\text{If } \vec{u} \neq \vec{0}_n, \text{ then } \vec{u} \circ \vec{u} > 0.}$$

The last two properties can be combined into one:

7. ***The Non-Degeneracy Property***

$$\vec{u} \circ \vec{u} > 0 \text{ if and only if } \vec{u} \neq \vec{0}_n,$$
$$\text{and } \vec{0}_n \circ \vec{0}_n = 0.$$

Pf. Assume $\vec{u} \circ \vec{u} > 0$ & write $\vec{u} = \langle u_1, u_2, \dots, u_n \rangle \in \mathbb{R}^n$.

Then
$$\vec{u} \circ \vec{u} = (u_1)^2 + (u_2)^2 + \cdots + (u_n)^2 > 0$$

$\Longleftrightarrow$ Then at least one of $u_1^2, \dots, u_n^2$ is not zero.

$\Longleftrightarrow$ at least one of $u_1, \dots, u_n$ is not zero

$\Longleftrightarrow$ $\vec{u} \neq \vec{0}$ $\square$

Pf $\vec{0}_n \circ \vec{0}_n = 0$ (excercise).

Example: Suppose we are told that $\vec{u}$ and $\vec{v}$ are two vectors from some $\mathbb{R}^n$ (which $\mathbb{R}^n$ is not really important). Suppose we were provided the information that $\|\vec{u}\| = 3$, $\|\vec{v}\| = 7$, and $\vec{u} \circ \vec{v} = 16$. Find $\|4\vec{u} - 9\vec{v}\|$.

Trick:

$$\|4\vec{u} - 9\vec{v}\|^2 = (4\vec{u} - 9\vec{v}) \circ (4\vec{u} - 9\vec{v})$$

$$= 16\, \vec{u} \circ \vec{u} - 36\,(\vec{u} \circ \vec{v}) - 36\, \vec{v} \circ \vec{u} + 81\,(\vec{v} \circ \vec{v})$$

$$= 16\,\|\vec{u}\|^2 - 72\,(\vec{u} \circ \vec{v}) + 81\,\|\vec{v}\|^2$$

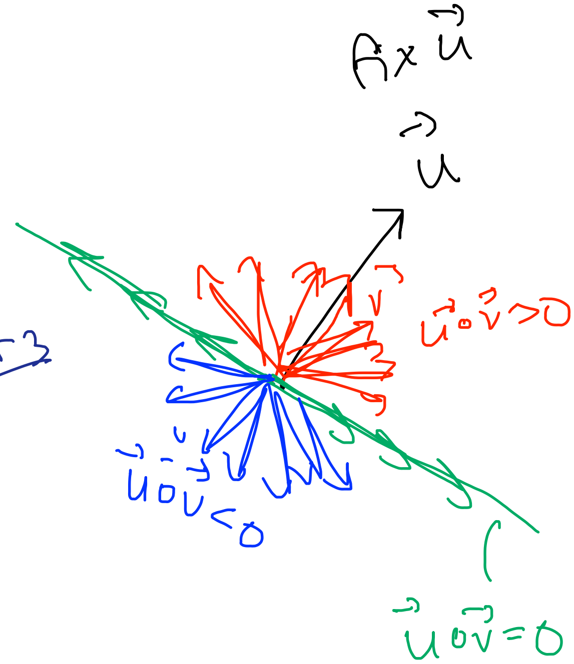
$$= 16(3)^2 - 72(16) + 81(7)^2$$

$$= \boxed{2961}$$

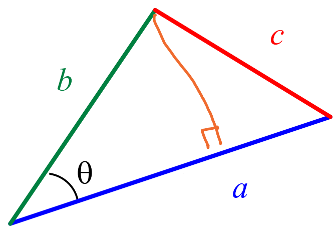
A Geometric Formulation for the Dot Product

$\mathbb{R}^n$ for $n=2$ or 3

$A \times \vec{u}$



The Law of Cosines:



LoC

$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$

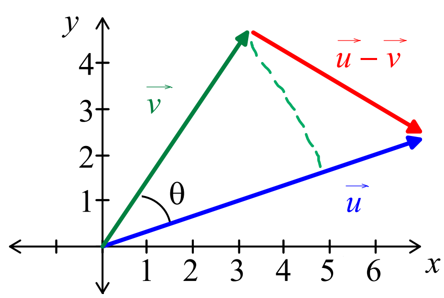
$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$

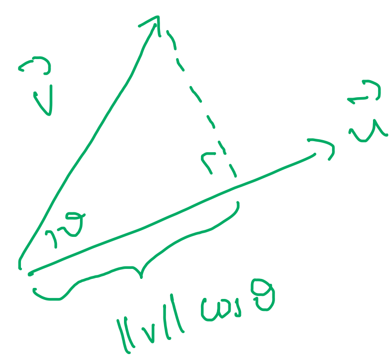
$(\vec{u}-\vec{v})\circ(\vec{u}-\vec{v}) =$

$\leftarrow =$

$\vec{u}\circ\vec{u} - \vec{u}\circ\vec{v} - \vec{v}\circ\vec{u} + \vec{v}\circ\vec{v}$

$\|\vec{u}\|^2 - 2(\vec{u}\circ\vec{v}) + \|\vec{v}\|^2$





The Triangle Formed by $\vec{v}$, $\vec{u}-\vec{v}$ and $\vec{u}$

cancelling: cancell $\|\vec{u}\|^2$ & $\|\vec{v}\|^2$

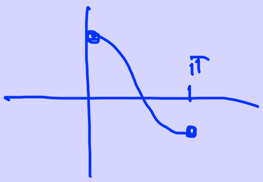
$-2(\vec{u}\circ\vec{v}) = -2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$

$\vec{u}\circ\vec{v} = \|\vec{u}\|\,\underline{\|\vec{v}\|\cos\theta}$

Definition/Theorem: If $\vec{u}$ and $\vec{v}$ are ***non-zero*** vectors in $\mathbb{R}^2$, then:

$$\boxed{\vec{u} \circ \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos(\theta),}$$

where θ is the angle formed by the vectors $\vec{u}$ and $\vec{v}$ in standard position. Thus, we can ***compute*** the angle θ between $\vec{u}$ and $\vec{v}$ by:



$$\boxed{\cos(\theta) = \frac{\vec{u} \circ \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|},} \longrightarrow \theta = \cos^{-1}\left(\frac{\vec{u} \circ \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}\right)$$

where $0 \leq \theta \leq \pi$. We will use the exact same formula for two vectors in $\mathbb{R}^3$.

Example: Let us consider the two vectors $\vec{u} = \langle 7, 4 \rangle$ and $\vec{v} = \langle -3, 2 \rangle$.

at home

Orthogonality in $\mathbb{R}^2$ *or* $\mathbb{R}^3$

Definition/Theorem: Two vectors $\vec{u}$ and $\vec{v} \in \mathbb{R}^2$ or $\mathbb{R}^3$ are ***perpendicular*** or ***orthogonal*** to each other ***if and only if*** $\boxed{\vec{u} \circ \vec{v} = 0.}$

Example: $\vec{u} = \langle 4, -2, 3 \rangle$ and $\vec{v} = \langle -3, 5, 7 \rangle$

Is $\vec{u}$ orthogonal to $\vec{v}$? $\boxed{\text{No}}$

$$\vec{u} \circ \vec{v} = \langle 4, -2, 3 \rangle \circ \langle -3, 5, 7 \rangle$$

$$= (4)(-3) + (-2)(5) + (3)(7)$$

$$= -12 - 10 + 21$$

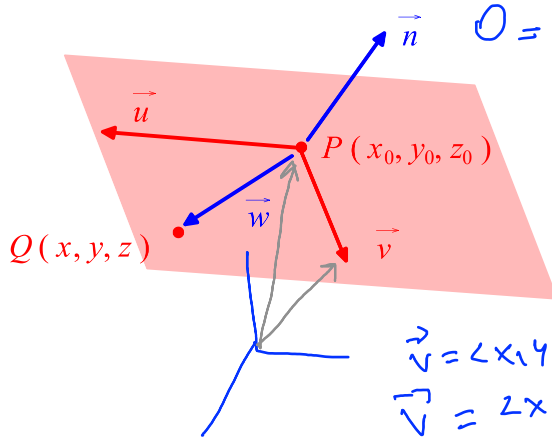
$$< 0$$

Revisiting The Cartesian Equation of a Plane

for all $\vec{v} \in \Pi$

$\vec{n} \circ \vec{v} = 0$

$0 = \langle a, b, c \rangle \circ \langle x - x_0, y - y_0, z - z_0 \rangle$



$\vec{v} = \langle x, y, z \rangle \notin \Pi$

$\vec{v} = \langle x, y, z \rangle - \vec{x}_p \in \Pi$

An Arbitrary Plane in Cartesian Space

$$ax + by + cz = d.$$

Comments recall 2D Law of Cosines

$-1 \le \cos\theta \le 1 \rightsquigarrow -1 \le \frac{\vec{u}\circ\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} \le 1$ $\left(\begin{matrix}\vec{u}\neq\vec{0}\\ \vec{v}\neq\vec{0}\end{matrix}\right)$

The Cauchy-Schwarz Inequality

$\rightarrow -\|u\|\,\|v\| \le \vec{u}\circ\vec{v} \le \|u\|\,\|v\|$

↳ statement about triangles in $\mathbb{R}^n$.

Theorem — The Cauchy-Schwarz Inequality: (DEEP)

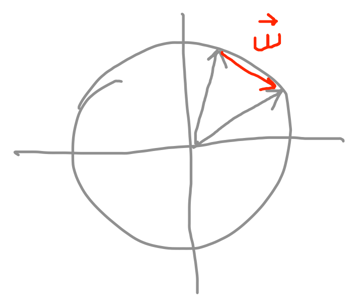
For any two vectors $\vec{u}$ and $\vec{v} \in \mathbb{R}^n$: $|\vec{u}\circ\vec{v}| \le \|\vec{u}\|\,\|\vec{v}\|$.

Proof: We will separate the proof into two cases:

Case 1: Suppose $\vec{u} = \vec{0}_n$ or $\vec{v} = \vec{0}_n$. Then both sides are 0, so the inequality is true. ✓

Case 2: Suppose now that $\vec{u} \neq \vec{0}_n$ and $\vec{v} \neq \vec{0}_n$.

Consider the vector $\vec{w} \in \mathbb{R}^n$ given by:



$$\vec{w} = \frac{\vec{u}}{\|\vec{u}\|} - \frac{\vec{v}}{\|\vec{v}\|}$$

Then

$$\|\vec{w}\|^2 = \vec{w}\circ\vec{w}$$
$$= \left(\frac{u}{\|u\|} - \frac{v}{\|v\|}\right)\circ\left(\frac{u}{\|u\|} - \frac{v}{\|v\|}\right)$$
$$= \frac{u\circ u}{\|u\|^2} - 2\frac{u\circ v}{\|u\|\,\|v\|} + \frac{v\circ v}{\|v\|^2}$$
$$= \underbrace{\frac{\|u\|^2}{\|u\|^2}}_{1} - 2\frac{u\circ v}{\|u\|\,\|v\|} + \underbrace{\frac{\|v\|^2}{\|v\|^2}}_{1}$$

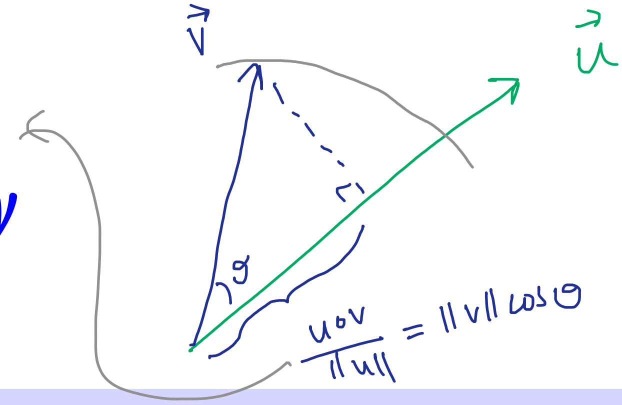
Observe $\|\vec{w}\|^2 \ge 0$

So: $2 - 2\frac{u\circ v}{\|u\|\,\|v\|} \ge 0$

$\iff u\circ v \le \|u\|\,\|v\|$. $\square$

$\frac{u \circ v}{\|u\|} < \|\vec{v}\|$

The Triangle Inequality

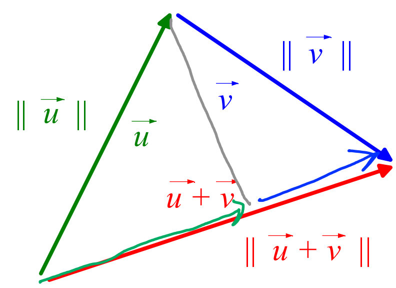


$\frac{u \circ v}{\|u\|} = \|v\| \cos\theta$

Theorem — The Triangle Inequality:

For any two vectors $\vec{u}$ and $\vec{v} \in \mathbb{R}^n$: $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$.

"=" when all lie on line.



The Triangle Inequality: $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$

Pf $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v}) \circ (\vec{u}+\vec{v})$

$= u \circ u + u \circ v + v \circ u + v \circ v$

$= \|u\|^2 + 2(\vec{u} \circ \vec{v}) + \|\vec{v}\|^2$

$\le \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$ (by Cauchy-Schwarz Inequality)

$= (\|\vec{u}\| + \|\vec{v}\|)^2$ (algebra)

Since $\|u+v\| \ge 0$, $\|u\|, \|v\| \ge 0$, taking square roots:

$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$. $\square$

Angles and Orthogonality

Definition: If $\vec{u}, \vec{v} \in \mathbb{R}^n$ are ***non-zero*** vectors, we define the ***angle*** θ ***between*** $\vec{u}$ ***and*** $\vec{v}$ by:

$$\boxed{\cos(\theta) = \frac{\vec{u} \circ \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|},}$$

where $0 \leq \theta \leq \pi$. Furthermore, we will say that $\vec{u}$ is ***orthogonal*** to $\vec{v}$ if $\boxed{\vec{u} \circ \vec{v} = 0.}$

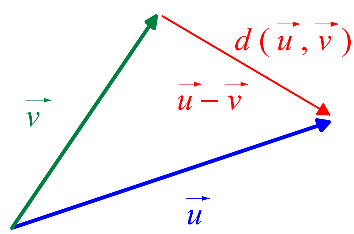
We will ***agree*** that the zero vector $\vec{0}_n$ is orthogonal to ***all*** vectors in $\mathbb{R}^n$.

Example: Find the angle θ between $\vec{u} = \langle 3, -7, 6, -4 \rangle$ and $\vec{v} = \langle 2, 1, -3, -2 \rangle$.

Distance Between Vectors

Definition: If $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ are vectors from $\mathbb{R}^n$, we define the ***distance between $\vec{u}$ and $\vec{v}$*** as:

$$\boxed{d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|}$$
$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}.$$



The Distance Between Two Vectors $\vec{u}$ and $\vec{v}$

Example: Let $\vec{u} = \langle 7, 3, -4, -2 \rangle$ and $\vec{v} = \langle -2, 0, 3, -4 \rangle$.

Theorem — Properties of Distances:

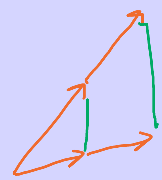
Let $\vec{u},\ \vec{v},\ \vec{w} \in \mathbb{R}^n$ and $k \in \mathbb{R}$. Then, we have the following properties:

1. *The Symmetric Property for Distances*

$$\boxed{d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u}).}$$

2. *The Homogeneity Property for Distances*

$$d(k\vec{u}, k\vec{v}) = |k| \cdot d(\vec{u}, \vec{v}).$$



3. *The Triangle Inequality for Distances*

$$d(\vec{u}, \vec{w}) \leq d(\vec{u}, \vec{v}) + d(\vec{v}, \vec{w}).$$

$$\|\vec{u} - \vec{w}\| \leq \|\vec{u} - \vec{v}\| + \|\vec{v} - \vec{w}\|$$

Pfs

do excersises!