1.4 SOE "System of Equations"

$$\begin{cases} x = 6 \\ y = -2 \\ z = 1 \end{cases} \longleftrightarrow \begin{aligned} x + 0y + 0z &= 6 \\ 0x + y + 0z &= -2 \\ 0x + 0y + z &= 1 \end{aligned} \longleftrightarrow \begin{array}{c c} \begin{array}{ccc} x & y & z \end{array} & \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

$A \mid b$

Identity Matrix I_3

$$\longleftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$$

$$\longleftrightarrow (A \mid b)$$

Goal of GJR (Gauss-Jordan Reduction Elimination Algorithm)

start

$$\underbrace{(A \mid b)}_{\text{def "augmented matrix"}} \xrightarrow[\text{EROs}]{\text{GJR}} (I^* \mid c^*) \quad \text{"continue w/ back substitution"}$$

or

$$(I_n \mid c^*) \quad \text{"done!"}$$

★★★★ **Theorem** The solution $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = c^* = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ to the original SOE

How to do it?

* use EROs,
* use "clever" alogrithm.

EROs Elementary Row Operations

1. swap any two rows $(R_i \leftrightarrow R_j)$
2. multiply entire row by a non-zero scalar $(cR_i \to R_i)$ $(c \neq 0)$
3. can replace any row by the sum of any two non-zero multiples of other rows.

$$(aR_i + bR_j \to R_i)$$

The Algorithm

Start: $(A|b) = \left(\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & \cdots & a_{mn} & b_m \end{array}\right)$

Step 1 turn column 1 into $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← start here & then go down

Using EROs.

Step 2 turn column 2 into $\begin{bmatrix} * \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ←start here & then go down

$*$ = ignore

Step 3 Continue in this way:

$$\left[\begin{array}{ccccc|c} 1 & * & * & * & & * \\ 0 & 1 & * & * & & * \\ 0 & 0 & 1 & * & & \vdots \\ \vdots & 0 & 0 & 1 & & \\ \vdots & \vdots & \vdots & 0 & & \\ 0 & 0 & 0 & 0 & \cdots \; 1 & * \end{array}\right]$$

Using EROs.

If we stop here: call this REF (ROW echelon form)

Important Can stop here & solve using using "back substitution")

Step 4 "go upwards & backwards"

make second to last column into $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$ ←start here & go upwards

Step 5: continue in this pattern until:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & & 0 & c_1 \\ 0 & 1 & 0 & & & c_2 \\ \vdots & 0 & 1 & \ddots & & \vdots \\ & \vdots & 0 & & & \vdots \\ 0 & 0 & 0 & & 1 & c_n \end{array}\right]$$

fin.

This last step is called RREF (Reduced Row Echelon Form)

Main Point $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ solves original SOEs!