1.5 Linear Systems and Linear Independence

Definition:

A linear system is called ***consistent*** if it has ***at least one*** solution.

A linear system is called ***inconsistent*** if it does ***not*** have any solutions. ϕ

ie can I find

$\underline{X} = \langle x_1, \ldots, x_n \rangle$

$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$ RHL Linear combination,

Theorem: Let $\vec{b} \in \mathbb{R}^m$ and let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors from $\mathbb{R}^m$. Then $\boxed{\vec{b} \in Span(S)}$ ***if and only if*** the system of equations corresponding to the ***augmented*** matrix:

$$A = \boxed{\left[\vec{v}_1 \;\; \vec{v}_2 \;\; \ldots \;\; \vec{v}_n \;\middle|\; \vec{b} \right]}$$

is ***consistent***.

m rows

$$x_1 \overset{\vec{v}_1}{\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}} + x_2 \overset{\vec{v}_2}{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \overset{\vec{b}}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

$\mathbb{R}^m$

m = rows

$v_1, \dots, v_n$

n = columns

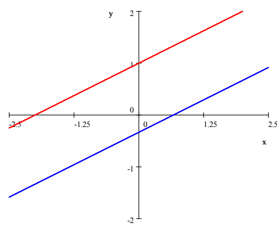
Definition: A linear system with m equations in n variables is called:

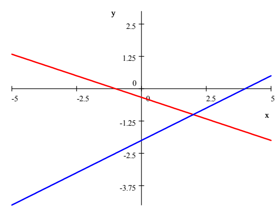
1. ***square*** if $\boxed{m = n.}$ square
2. ***underdetermined*** if $\boxed{m < n}$. wide
3. ***overdetermined*** if $m > n$. thin

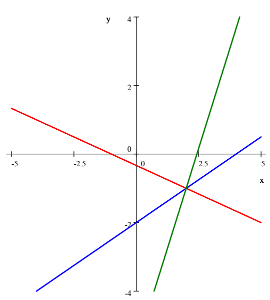
wide $\rightsquigarrow$ "generally" lots of sol

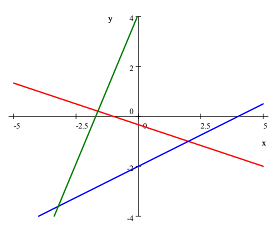
thin $\rightsquigarrow$ hard to have sol

Geometric Interpretation in $\mathbb{R}^2$ *and* $\mathbb{R}^3$

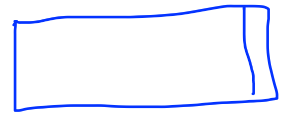


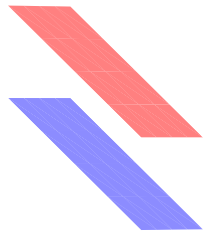




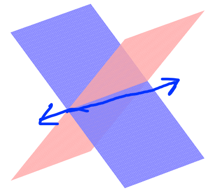


2 eq in 3 unknowns





$\emptyset$



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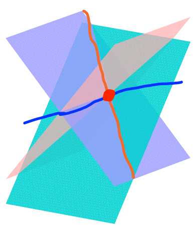

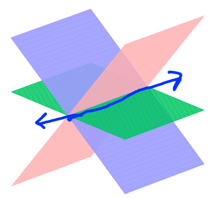

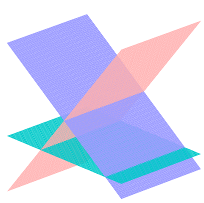

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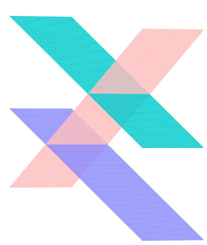


$\emptyset$

Example: Let us investigate the system:

$$\begin{array}{rcrcrcr} 2x & + & 3y & - & z & = & 5 \\ 5x & + & 4y & - & 3z & = & 7 \\ -7x & + & 7y & + & 6z & = & 10 \end{array}$$

∞ly many solutions!

Homogeneous Systems

Definition: A ***homogeneous*** system of m equations in n unknowns is a system of linear equations where the right side of the equations consists entirely of zeros. In other words, the augmented matrix has the form:

ie $\vec{b} = \vec{0}$

$$\left[A \middle| \vec{0}_m \right],$$

$\boxed{A\vec{x} = \vec{0}}$ HSOE

where A is an $m \times n$ matrix. If the right side $\vec{b}$ is not the zero vector, we call the system ***non-homogeneous***.

Clearly, $\vec{x} = \vec{0}_n = \langle 0, 0, \ldots, 0 \rangle$ is a solution to the homogeneous system. We call this the ***trivial solution*** to a homogeneous system, and any other solution is called a ***non-trivial solution***.

$\vec{v}_1$ $\vec{v}_2$

$$0 \begin{bmatrix} \\ \\ \end{bmatrix} + 0 \begin{bmatrix} \\ \\ \end{bmatrix} + \cdots + 0 \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

HSOE always has a solution $(\vec{x} = \vec{0})$.

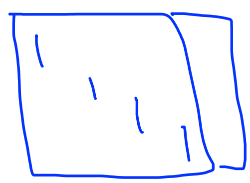
When do we get an Infinite Number of Solutions? (or just one sol = $\vec{0}$)

Theorem: A homogeneous system has an infinite number of solutions (and hence, non-trivial solutions) $\boxed{\textit{if and only if}}$ the rref of A has free variables.

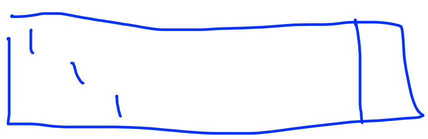
↳ entire row of zeros!

$[0\ 0 \cdots 0 \mid 0]$

What shape of system always has a free variable?



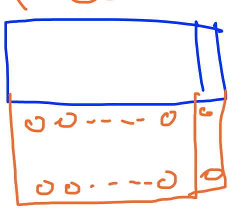
Square? No.



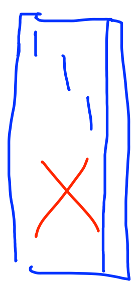
wide? No? Free!

make in square

add rows of 0s



∞ly many sol



thin? No!

Summary: "wide"

Theorem: An ***underdetermined homogeneous system*** always has an ***infinite*** number of solutions. In other words, a homogeneous system with ***more variables than equations*** has an infinite number of solutions.

Example:

$$\left[\begin{array}{rrrr|r} 4 & -8 & 3 & 9 & 6 \\ 3 & -6 & -4 & 13 & 17 \\ -2 & 4 & 3 & -9 & -12 \end{array}\right]$$

RREF

$$\rightarrow \left[\begin{array}{rrrr|r} 1 & -2 & 0 & 3 & 3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \infty\text{ly many sol.}$$

Matrix Products

Set-up:

Identify a vector with a column matrix:

$$\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Partition a matrix into ***columns:***

$$A = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix},$$

Definition — Matrix Product:

If $A = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix}$ is an $m \times n$ matrix and $\vec{x} \in \mathbb{R}^n$, we define the ***matrix product*** $A\vec{x}$ to be the linear combination:

$$\boxed{A\vec{x}} = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix} \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}^{\text{new}}$$

did this

$$= \boxed{x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n}.$$

Notice that since each column is an $m \times 1$ matrix, the matrix product is again an $m \times 1$ matrix. Thus, $A\vec{x}$ is a ***linear combination*** of the columns of A with coefficients from $\vec{x}$, and so $A\vec{x} \in \mathbb{R}^m$.

$$\begin{array}{c} \\ m \text{ rows} \left\{ \vphantom{\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}} \right. \end{array} \overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ & & \vdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}^{n \text{ cols}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

dot product of row • column

$m \times \boxed{n}$ $\quad \boxed{n} \times 1$

$=$

get $m \times 1$

*key = inside dimensions MUST match!

Example:

$$3 \times \boxed{4} \overset{=}{\checkmark} \boxed{4} \times 1$$

$$\begin{array}{cc} & \begin{matrix} 1 & \;\;2 & \;\;3 & \;\;4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 7 & -1 & -2 & 6 \\ -2 & 5 & 3 & -4 \\ 8 & 3 & -5 & 1 \end{bmatrix} \end{array} \begin{bmatrix} 4 \\ -2 \\ 3 \\ 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 54 \\ -29 \\ 16 \end{bmatrix}}_{3 \times 1}$$

★ Pro tip: Check dimensions before jumping into calculation!

Theorem — Properties of Matrix Multiplication:

For all $m \times n$ matrices A, for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, and for all $k \in \mathbb{R}$, matrix multiplication enjoys the following properties:

The Additivity Property	$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}.$
The Homogeneity Property	$A(k\vec{x}) = k(A\vec{x}).$

Pf (HW) excercise. $\square$

The Matrix Product Form of Linear Systems

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{b}.$$

We formed the augmented matrix $\left[\, \vec{v}_1 \; \vec{v}_2 \; \cdots \; \vec{v}_n \mid \vec{b} \,\right]$ and looked at its rref.

Alternative way:

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}.$$

Matrix Equation:

$$A\vec{x} = \vec{b}$$

Rephrase Consistency Requirement for Membership in a Span

Theorem: Suppose that $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors from $\mathbb{R}^m$, and $\vec{b} \in \mathbb{R}^m$. Let us form the $m \times n$ matrix:

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}.$$

Then, $\vec{b} \in \mathrm{Span}(S)$ ***if and only if*** the matrix equation $A\vec{x} = \vec{b}$ is ***consistent***.

Major Concept: Linear Dependence and Independence

$\vec{v}_i \in \mathbb{R}^m$

Definition: A set of vectors $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ from $\mathbb{R}^m$ is ***linearly dependent*** if we can find a ***non-trivial solution*** $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{R}^n$, where at least one component is ***not zero***, to the vector equation:

$$* \quad x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}_m. \quad \text{(HSOE)}$$

We will call this equation the ***dependence test equation*** for S. An equation of this form where at least one coefficient is ***not zero*** will be referred to as a ***dependence equation***. Thus, for S to be linearly dependent, we must find a non-trivial solution $\vec{x}$ to the homogeneous system:

$$(A \mid \vec{0}) \qquad \boxed{A\vec{x} = \vec{0}_m,} \qquad \text{use GJR}$$

∞ly many sol iff row 0s iff free variables.

where $A = \left[\, \vec{v}_1 \mid \vec{v}_2 \mid \ldots \mid \vec{v}_n \,\right]$ is the matrix with the vectors $\vec{v}_1$, $\vec{v}_2$, …, $\vec{v}_n$ as its ***columns***. This is equivalent to the presence of a ***free variable*** in the rref of the matrix A.

Recall HSOE always have $\vec{0}$ as a sol. (ie consistent)

DE $\Rightarrow$ HSOE $\Rightarrow$ ∞ly many sol $\Rightarrow$ $(0\ 0 \ldots 0)$

However, if only the trivial solution $\vec{x} = \vec{0}_n$ exists for the dependence test equation, we say that S is ***linearly independent***.

We often drop the adjective "linearly" and simply say that a set S is ***dependent*** or ***independent***.

LI = linearly independent

DTE

$\vec{v}_1, \dots, \vec{v}_n$ are LI iff $[x_1 \vec{v} + x_2 \vec{v}_2 + \dots + x_n \vec{v}_n = \vec{0}]$

$\Downarrow$

[every $x_i = 0$ ie]

$x_1 = x_2 = x_3 = \dots = x_n = 0$

Example: The standard basis $S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m\}$.

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ldots$

DTE

$x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\vec{x} = \vec{0}}_{\text{DTE}}$ only sol so LI.

Example: Suppose that $\vec{v}_1 = \langle 4, -5, 3, -2 \rangle$, $\vec{v}_2 = \langle 7, -6, 2, -4 \rangle$ and $\vec{v}_3 = \langle -1, -7, 9, 2 \rangle$. Determine LD or LI.

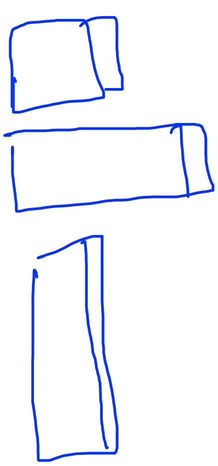
DTE:

$[x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 = \vec{0}]$

$\iff x_1 \begin{bmatrix} 4 \\ -5 \\ 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ -6 \\ 2 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -7 \\ 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

$\iff \left[\begin{array}{ccc|c} 4 & 7 & -1 & 0 \\ -5 & -6 & -7 & 0 \\ 3 & 2 & 9 & 0 \\ -2 & -4 & 2 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{\text{GJR}} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

two rows of 0s so $\boxed{\text{LD}}$



RREF

- $x_1 + 5x_3 = 0$
- $x_2 - 3x_3 = 0$
- $x_3 = \text{free}$

$\mathbb{X} = \vec{x} = \left\{ \begin{bmatrix} -5t \\ 3t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$

$= \{ \langle -5t, 3t, t \rangle : t \in \mathbb{R} \}$.

$\boxed{t=1}$ DE $\boxed{-5\vec{v}_1 + 3\vec{v}_2 + \vec{v}_3 = \vec{0}}$ write $\vec{v}_3$ as LC of $\vec{v}_1$ & $\vec{v}_2$:

Classifying Small Sets of Vectors

$$\boxed{\vec{v}_3 = 5\vec{v}_1 - 3\vec{v}_2}$$

Theorem: Any set $S = \left\{\vec{0}_n, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right\} \subset \mathbb{R}^m$ containing $\vec{0}_m$ is a ***dependent*** set.

A set w/ $\vec{0}$ is LD.

$$1 \cdot \vec{0} + 0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_n = \vec{0}$$ ✓

$$\vec{x} = \langle 1, 0, 0, \ldots, 0 \rangle \neq \vec{0}.$$

Theorem: A set $S = \{\vec{v}\}$ consisting of a single ***non-zero*** vector $\vec{v} \in \mathbb{R}^m$ is ***independent***.

When is $S = \{\vec{u}, \vec{v}\}$ linearly dependent / independent?

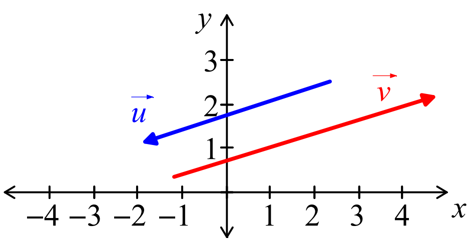
$$x_1\vec{u} + x_2\vec{v} = \vec{0} \iff \vec{v} = \left(\frac{-x_1}{x_2}\right)\vec{u}$$

assuming $x_2 \neq 0$.

$$\iff \vec{v} \parallel \vec{u}$$

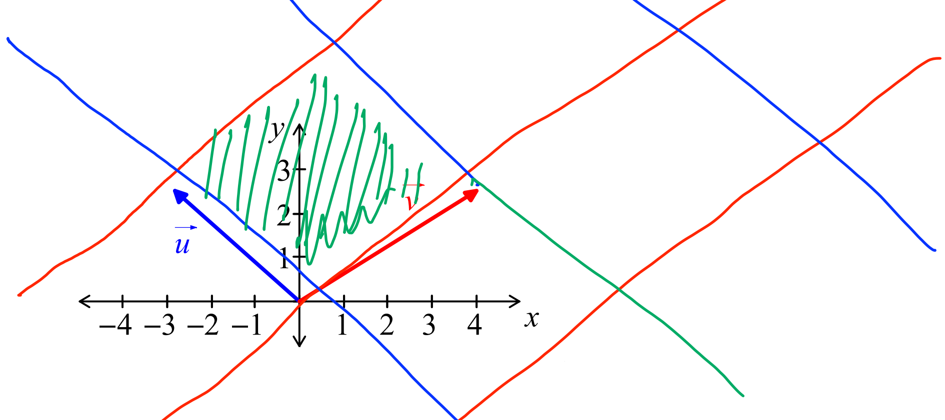
$\iff \vec{v}$ parallel to $\vec{u}$

Theorem: A set $S = \{\vec{u}, \vec{v}\}$ consisting of ***two*** vectors from $\mathbb{R}^m$ is ***dependent*** if and only if $\vec{u}$ and $\vec{v}$ are ***parallel*** to each other.



Two Dependent

(Parallel) Vectors



Two Independent

(Non-Parallel) Vectors

Example: $\{\langle 15, -10, 20, -25\rangle, \langle -9, 6, -12, 15\rangle\}$

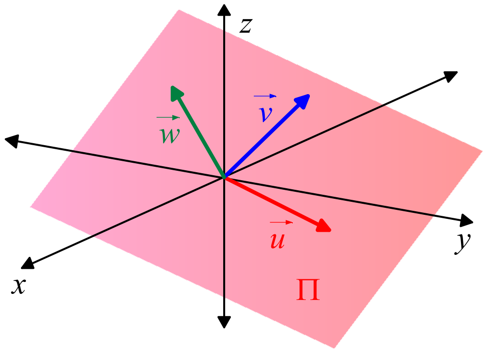
$5\langle 3, -2, 4, -5\rangle \qquad -3\langle 3, -2, 4, -5\rangle$

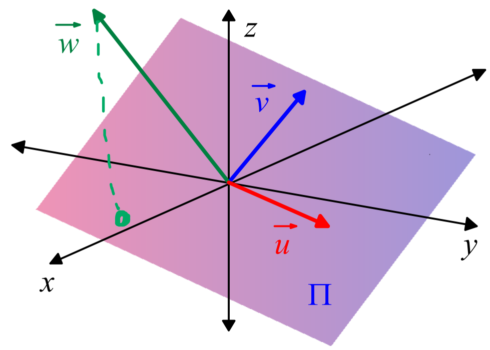
LD

$\langle 15, -10, 20, -25\rangle = \frac{-5}{3}\langle -9, 6, -12, 15\rangle$

When is $S = \{\vec{u}, \vec{v}, \vec{w}\}$ linearly dependent / independent?

Theorem: A set $S = \{\vec{u}, \vec{v}, \vec{w}\}$ consisting of ***three*** non-zero vectors from $\mathbb{R}^m$ is ***dependent*** if and only if $\vec{u}$, $\vec{v}$ and $\vec{w}$ are ***coplanar***.





Three Dependent (Non-Parallel) Vectors where	Three Independent (Non-Parallel) Vectors where
$\vec{w} \in Span(\{\vec{u}, \vec{v}\}) = \Pi$	$\vec{w} \notin Span(\{\vec{u}, \vec{v}\}) = \Pi$

$$= \{x_1 \vec{u} + x_2 \vec{v} \mid x_1, x_2 \in \mathbb{R}\} \quad \text{all}$$

Example:

$$S = \{\langle 2, -3, 4\rangle, \langle 5, 3, -6\rangle, \langle -4, -2, 7\rangle\}.$$

graph tough to tell (even w/ computer)

RREF says $\vec{x} = \vec{0}$ only sol. LI

Another Way to Think of Linear Dependence/Independence

Theorem: Suppose that $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a set of non-zero vectors from some $\mathbb{R}^m$, and S contains at least two vectors. Then: S is linearly dependent ***if and only if*** at least one vector $\vec{v}_i$ from S can be expressed as a linear combination of the other vectors in S.

Guaranteed Dependence

If the vectors are from $\mathbb{R}^n$, what is the minimum number of vectors required to produce an underdetermined system?

Theorem: A set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ of m vectors from $\mathbb{R}^n$ is automatically linearly ***dependent*** if $\boxed{m > n}$.

Nice, easy test to know & use!

Example:

$$S = \left\{\left\langle 5, -3, 0, 2\right\rangle, \langle 2, -7, 3, -8\rangle, \left\langle 1, 0, -2, 4\right\rangle, \langle -5, 1, 6, -3\rangle, \langle -2, 5, 1, 6\rangle\right\}$$

5 vec > dim = 4

LD