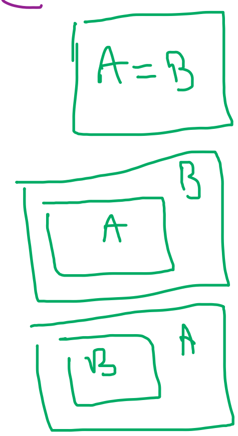
1.6

Thm.
- $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subset \mathbb{R}^m$
- $S' = \{k_1\vec{v}_1, k_2\vec{v}_2, \dots, k_n\vec{v}_n\} \subset \mathbb{R}^m$
- ($k_i \neq 0$ scalar for all $i = 1, \dots, n$)

Then $\boxed{\text{Span}(S) = \text{Span}(S')}$



Pf. ($\subseteq$) WTS: $\overset{A}{\text{Span}(S)} \subseteq \overset{B}{\text{Span}(S')}$

ie $\vec{w} \in \text{Span}(S) \implies \vec{w} \in \text{Span}(S')$.

Let $\vec{w} \in \text{Span}(S)$. WTS: $\vec{w} \in \text{Span}(S')$

Then there exists $x_1, x_2, \dots, x_n \in \mathbb{R}$ so that

$$\vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n$$

$$= x_1\left(\frac{k_1}{k_1}\right)\vec{v}_1 + x_2\left(\frac{k_2}{k_2}\right)\vec{v}_2 + \dots + x_n\left(\frac{k_n}{k_n}\right)\vec{v}_n$$

$$= \left[\frac{x_1}{k_1}\right](k_1\vec{v}_1) + \left(\frac{x_2}{k_2}\right)(k_2\vec{v}_2) + \dots + \left(\frac{x_n}{k_n}\right)(k_n\vec{v}_n)$$

Thus, $\vec{w} \in \text{Span}(S')$.

$(\supseteq)$ WTS: $\mathrm{Span}(S') \subseteq \mathrm{Span}(S)$

ie $\vec{w} \in \mathrm{Span}(S') \Longrightarrow \vec{w} \in \mathrm{Span}(S)$.

(Exercise.) $\square$

Thm Equality of Spans Theorem

- $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subset \mathbb{R}^k$
- $S' = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\} \subset \mathbb{R}^k$

$\boxed{\mathrm{Span}(S) = \mathrm{Span}(S')}$ $\boxed{\text{iff}}$ $\left(\forall \vec{v}_i,\ \vec{v}_i \text{ is LC of } \vec{w}_1, \ldots, \vec{w}_m\right)^P$

&

$\left(\forall \vec{w}_j,\ \vec{w}_j \text{ is LC of } \vec{v}_1, \ldots, \vec{v}_n\right)^Q$

Pf $(\Longrightarrow)$ Assume $\mathrm{Span}(S) = \mathrm{Span}(S')$.

- For each $\vec{v}_i \in S \subseteq \text{Span}(S) = \text{Span}(S')$, it is a LC of vectors in $S' = \{\vec{w}_1, \dots, \vec{w}_m\}$.
- Similarly, for each $\vec{w}_j \in S' \subseteq \text{Span}(S') = \text{Span}(S)$, it is a LC of vectors in $S = \{\vec{v}_1, \dots, \vec{v}_n\}$.

($\Longleftarrow$) Assume P and Q are true.

<u>WTS</u> $\text{span}(S) = \text{Span}(S')$.

(1) <u>NTS</u> $\text{Span}(S) \subseteq \text{Span}(S')$

(2) <u>NTS</u> $\text{Span}(S') \subseteq \text{Span}(S)$.

(1) Assume $\vec{w} \in \text{Span}(S)$. By definition of Span,

$$(*) \quad \vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \quad (\text{some } x_i \in \mathbb{R})$$

Next, for each $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ condition P says that there exists $a_{1i}, a_{2i}, a_{3i}, \dots, a_{mi} \in \mathbb{R}$

so that

$$(**) \qquad \vec{v}_i = a_{1i}\vec{w}_1 + a_{2i}\vec{w}_2 + \cdots + a_{mi}\vec{w}_m$$

for $i = 1, 2, \ldots, n$. This is called double-index notation.

So substituting $(**)$ into $(*)$:

$$\vec{w} = x_1\Big[\underbrace{a_{11}\vec{w}_1 + a_{21}\vec{w}_2 + \cdots + a_{m1}\vec{w}_m}_{\vec{v}_1}\Big]$$

$$+ x_2\Big[\underbrace{a_{12}\vec{w}_1 + a_{22}\vec{w}_2 + \cdots + a_{m2}\vec{w}_m}_{\vec{v}_2}\Big]$$

$$+ \cdots +$$

$$x_n\Big[a_{1n}\vec{w}_1 + a_{2n}\vec{w}_2 + \cdots + a_{mn}\vec{w}_m\Big]$$

$$= \Big\{\overbrace{x_1a_{11} + x_2a_{12} + x_3a_{13} + \cdots + x_na_{1n}}^{c_1 \,=}\Big\}\vec{w}_1$$

$$+ \Big\{\overbrace{x_1a_{21} + x_2a_{22} + x_3a_{23} + \cdots + x_na_{2n}}^{=\, c_2}\Big\}\vec{w}_2$$

$$+ \cdots +$$

$\Big\{$ similar $\Big\}\vec{w}_m$

$\vec{w} = c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_m\vec{w}_m$, where $c_1, \dots, c_m \in \mathbb{R}$.

This says $\vec{w} \in \mathrm{Span}(S')$.

(2) To prove this step, we follow a similar argument. $\square$

Comments

* this theorem is theoretically useful but not practically.
  ↳ need to solve $n \times m$ SOE!
  ↳ usually a huge task!!

Ex Cont.

$$\vec{w}_2 = x_1\vec{v}_1 + x_2\vec{v}_2 \longrightarrow \begin{array}{c} \\ \left[\begin{array}{cc|c} 3 & 2 & -4 \\ -5 & -4 & 14 \\ 2 & 1 & 1 \\ -4 & -2 & -2 \end{array}\right] \\ \vec{v}_1 \quad \vec{v}_2 \quad \text{new } \vec{w}_2 \end{array} \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -11 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

6 col1 − 11 col2 = col3

Magic $\boxed{6\vec{v}_1 - 11\vec{v}_2 = \vec{w}_2}$

- Is $\vec{w}_3$ also a LC of $\vec{v}_1, \vec{v}_2$?

$$\begin{array}{c}\left[\begin{array}{cc|c} 3 & 2 & 1 \\ -5 & -4 & 3 \\ 2 & 1 & 3 \\ -4 & -2 & -6 \end{array}\right] \\ \vec{w}_3 \end{array} \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \overset{\text{Magic}}{\boxed{\vec{w}_3 = 5\vec{v}_1 - 7\vec{v}_2}}$$

So $\vec{w}_1, \vec{w}_2, \vec{w}_3 \in \text{Span}(\vec{v}_1, \vec{v}_2)$. #

Thm (Dependent Sets from Span Thm)

$S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

$L = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\} \subset \text{Span}(S)$.

If $m > n$ then L is LD.

Pf Let $\vec{u}_i \in \text{Span}(S)$ $i = 1, 2, \dots, m$.

(Double-index notation)

Def of Span: $\vec{u}_i = \sum_{j=1}^{n} a_{ij}\vec{v}_j = a_{i1}\vec{v}_1 + a_{i2}\vec{v}_2 + \dots + a_{in}\vec{v}_n$

(for some $a_{ij} \in \mathbb{R}$)

So:

$$(*)\quad \begin{cases} \vec{u}_1 = a_{11}\vec{v}_1 + a_{12}\vec{v}_2 + \cdots + a_{1n}\vec{v}_n \\ \vec{u}_2 = a_{21}\vec{v}_1 \quad a_{22}\vec{v}_2 + \cdots + a_{2n}\vec{v}_n \\ \vdots \\ \vec{u}_m = a_{m1}\vec{v}_1 + a_{m2}\vec{v}_2 + \cdots + a_{mn}\vec{v}_n \end{cases}$$

- Next, our goal is to prove that L is LD. So let's set up the DTE:

DTE $\quad c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_m\vec{u}_m = \vec{0} \qquad (**)$

- if we can find a non-zero sol to this $\vec{c} = \langle c_1, c_2, \dots, c_m\rangle$ then L is LD.

- Combining $(*)$ & $(**)$ into one one eq (ie rewriting $(**)$):

$$c_1\left[a_{11}\vec{v}_1 + a_{12}\vec{v}_2 + \cdots + a_{1n}\vec{v}_n\right] + c_2\left[a_{21}\vec{v}_1 + a_{22}\vec{v}_2 + \cdots + a_{2n}\vec{v}_n\right]$$

$$+ \cdots + c_m\left[a_{m1}\vec{v}_1 + a_{m2}\vec{v}_2 + \cdots + c_{mn}\vec{v}_n\right] = \vec{0}$$

re-group:

$$\left[c_1a_{11} + c_2a_{21} + \cdots + c_m a_{m1}\right]\vec{v}_1$$

$$+ \left[c_1a_{12} + c_2a_{22} + \cdots + c_m a_{m2}\right]\vec{v}_2$$

$$+ \cdots +$$

$$\left[c_1a_{1n} + c_2a_{2n} + \cdots + c_m a_{mn}\right]\vec{v}_n = \vec{0}$$

True if we set all coefficients $= 0$:

HSOE : $A\vec{c} = \vec{0}$

$$\begin{cases} c_1 a_{11} + c_2 a_{21} + \cdots + c_m a_{m1} = 0 \\ c_1 a_{12} + c_2 a_{22} + \cdots + c_m a_{m2} = 0 \\ \quad \vdots \\ c_1 a_{1n} + c_2 a_{2n} + \cdots + c_m a_{mn} = 0 \end{cases}$$

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ & & \vdots & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$n \times m$

Now use $m > n$

$\boxed{\text{wide}}$

* must have an entire row of 0s in RREF

* therefore GJRA says have ∞ly many sol!

This says $\vec{c} \neq \vec{0} \implies L$ is LD.

$\square$