1.6 has deepest theorems in the chapter.

# 1.6 Independent Sets versus Spanning Sets

The concepts of ***Spans*** and ***independence*** are two of the most important concepts in Linear Algebra.

We will see Theorems connecting Spans of sets of vectors, and linearly independent or dependent sets.

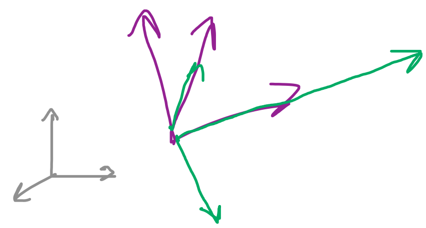
Goal take Span$(\{\vec{v}_1, \ldots, \vec{v}_n\}$

& reduce the # until we have

the smallest linearly independent set

with the same span.

# *Equality of Spans*



***Theorem:*** Let $\boxed{S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \mathbb{R}^m,}$ and $k_1$, $k_2, \ldots, k_n \in \mathbb{R}$ a list of $n$ ***non-zero scalars.*** Let us form a new set: $S' = \{k_1\vec{v}_1, k_2\vec{v}_2, \ldots, k_n\vec{v}_n\}$. Then: $\boxed{Span(S) = Span(S').}$

$$
\begin{aligned}
& c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \\
= {} & c_1\frac{k_1}{k_1}\vec{v}_1 + c_2\frac{k_2}{k_2}\vec{v}_2 + \cdots + c_n\frac{k_n}{k_n}\vec{v}_n \\
= {} & \frac{c_1}{k_1}(k_1\vec{v}_1) + \frac{c_2}{k_2}(k_2\vec{v}_2) + \cdots + \frac{c_n}{k_n}(k_n\vec{v}_n),
\end{aligned}
$$

$$
\begin{aligned}
& c_1(k_1\vec{v}_1) + c_2(k_2\vec{v}_2) + \cdots + c_n(k_n\vec{v}_n) \\
= {} & (c_1k_1)\vec{v}_1 + (c_2k_2)\vec{v}_2 + \cdots + (c_nk_n)\vec{v}_n,
\end{aligned}
$$

## *Example:*

$$S = \left\{ \begin{array}{c} \overset{\vec{v}_1}{\langle 3, -2, 5, 7, 4 \rangle}, \overset{\vec{v}_2}{\langle 2, -5, 3, 6, 0 \rangle}, \\ \underset{\vec{v}_3}{\langle -1, 0, 4, -3, 2 \rangle} \end{array} \right\}$$

$\vec{v}_2$ $\qquad\qquad\qquad$ $\vec{v}_3$

$3\langle 2, -5, 3, 6, 0\rangle$ $\qquad$ $5\langle -1, 0, 4, -3, 2\rangle$

$$S' = \left\{ \begin{array}{c} \langle 6, -15, 9, 18, 0 \rangle, \langle -5, 0, 20, -15, 10 \rangle, \\ \langle -6, 4, -10, -14, -8 \rangle \end{array} \right\}$$

$-2\langle 3, -2, 5, 7, 4\rangle$

$\vec{v}_1$

Is $Span(S) = Span(S')$? Yes

$S \,\&\, S'$ can have different # of vectors.

***Theorem — The Equality of Spans Theorem:***

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and $S^/ = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ be two sets of vectors from some Euclidean space $\boxed{\mathbb{R}^k}$. Then: $Span(S) = Span(S^/)$ ***if and only if*** every $\vec{v}_i$ can be written as a linear combination of the $\vec{w}_1$ through $\vec{w}_m$, ***and*** every $\vec{w}_j$ can also be written as a linear combination of the $\vec{v}_1$ through $\vec{v}_n$.

***Proof:***

$(\Rightarrow)$ $Span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\})$ includes $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ themselves.

$(\Leftarrow)$ Now, suppose that every $\vec{v}_i$ can be written as a linear combination of the $\vec{w}_1$ through $\vec{w}_m$, ***and*** every $\vec{w}_j$ can also be written as a linear combination of the $\vec{v}_1$ through $\vec{v}_n$.

Think of the linear combination:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

$$
\begin{aligned}
\vec{v}_1 &= a_{1,1}\vec{w}_1 + a_{1,2}\vec{w}_2 + \cdots + a_{1,m}\vec{w}_m, \\
\vec{v}_2 &= a_{2,1}\vec{w}_1 + a_{2,2}\vec{w}_2 + \cdots + a_{2,m}\vec{w}_m, \ldots\ldots \\
\vec{v}_n &= a_{n,1}\vec{w}_1 + a_{n,2}\vec{w}_2 + \cdots + a_{n,m}\vec{w}_m,
\end{aligned}
$$

$$
\begin{aligned}
& c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \\
= \; & c_1(a_{1,1}\vec{w}_1 + a_{1,2}\vec{w}_2 + \cdots + a_{1.m}\vec{w}_m) + \\
& c_2(a_{2,1}\vec{w}_1 + a_{2,2}\vec{w}_2 + \cdots + a_{2.m}\vec{w}_m) + \cdots + \\
& c_n(a_{n,1}\vec{w}_1 + a_{n,2}\vec{w}_2 + \cdots + a_{n.m}\vec{w}_m).
\end{aligned}
$$

$$
\begin{aligned}
& c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \\
= \; & c_1a_{1,1}\vec{w}_1 + c_1a_{1,2}\vec{w}_2 + \cdots + c_1a_{1.m}\vec{w}_m + \\
& c_2a_{2,1}\vec{w}_1 + c_2a_{2,2}\vec{w}_2 + \cdots + c_2a_{2.m}\vec{w}_m + \cdots + \\
& c_na_{n,1}\vec{w}_1 + c_na_{n,2}\vec{w}_2 + \cdots + c_na_{n.m}\vec{w}_m \\
= \; & (c_1a_{1,1} + c_2a_{2,1} + \cdots + c_na_{n,1})\vec{w}_1 + \\
& (c_1a_{1,2} + c_2a_{2,2} + \cdots + c_na_{n,2})\vec{w}_2 + \cdots + \\
& (c_1a_{1,m} + c_2a_{2,m} + \cdots + c_na_{n,m})\vec{w}_m.
\end{aligned}
$$

***Example:*** in $\mathbb{R}^4$

$$Span(\{\underset{\vec{v}_1}{\langle 3,-5,2,-4\rangle}, \underset{\vec{v}_2}{\langle 2,-4,1,-2\rangle}\})$$

*vs.*

$$Span(\underset{\vec{w}_1}{\langle 8,-14,5,-10\rangle}, \underset{\vec{w}_2}{\langle -4,14,1,-2\rangle}, \underset{\vec{w}_3}{\langle 1,3,3,-6\rangle}).$$

- $Span(\{\vec{v}_1, \vec{v}_2\}) = Span(\{\vec{w}_1, \vec{w}_2, \vec{w}_3\})$ ?
- start $\langle 8\ -14\ 5\ -10\rangle = x_1 \vec{v}_1 + x_2 \vec{v}_2$

iff $\underset{A}{\left[\begin{array}{cc|c} 3 & 2 & 8 \\ -5 & -4 & -14 \\ 2 & 1 & 5 \\ -4 & -2 & -10 \end{array}\right]} \xrightarrow{RREF} \underset{R}{\left[\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]} \rightarrow \infty$ ly many sol.

$2(\text{col } 1) + (\text{col } 2) = \text{col } 3$

Magic $\boxed{2\vec{v}_1 + \vec{v}_2 = \vec{w}_1}$

Q: when can we remove vectors from $S'$ but maintain the same span as $S'$.

**Theorem — The Elimination Theorem:**

Suppose that $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a ***linearly dependent*** set of vectors from $\mathbb{R}^m$, and $\vec{v}_n = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{n-1}\vec{v}_{n-1}$. Then:

$$\boxed{Span(S) = Span(S - \{\vec{v}_n\})}.$$

In other words, we can ***eliminate*** $\vec{v}_n$ from $S$ and still maintain the ***same Span***.

More generally, if $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}_m$, where ***none*** of the coefficients in this dependence equation is $0$, then:

$$Span(S) = Span(S - \{\vec{v}_i\}), \quad (c_i \neq 0)$$

for all $i = 1..n$. Pf Exercise. □

Ex let $S' = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$

let $\vec{v} \in Span(S')$.

$$\vec{v} = x_1\vec{w}_1 + x_2\vec{w}_2 + x_3\vec{w}_3$$

Ask Can we shorten to LC of $\vec{v}_1, \vec{v}_2$?

$$\vec{v} = x_1[2\vec{v}_1 + \vec{v}_2] + x_2[6\vec{v}_1 - 11\vec{v}_2] + x_3[5\vec{v}_1 - 7\vec{v}_2]$$

$$= \bigcirc\,\vec{v}_1 + \bigcirc\,\vec{v}_2$$

***Example:*** Let:

$$S = \left\{ \begin{matrix} \langle 3, 3, 5, 2, 4 \rangle, \langle 3, 2, 0, -1, 6 \rangle, \\ \langle 12, 12, 20, 8, 16 \rangle, \langle 0, 1, 5, 3, -2 \rangle \end{matrix} \right\},$$

$\mathrm{Span}(S) = \mathrm{Span}(\{\vec{v}_1, \vec{v}_2\})$. Final analysis ☺

and let us call these vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$, in that order.

Observe: • $\vec{v}_3 = 4\vec{v}_1$

so $\vec{v}_3 \in \mathrm{Span}(S \setminus \{\vec{v}_3\}) = \mathrm{Span}(v_1, v_2, v_4)$

so $\mathrm{Span}(S) = \mathrm{Span}(v_1, v_2, v_4)$.

• Observe: $\vec{v}_4 = \vec{v}_1 - \vec{v}_2$

so $\vec{v}_4 \in \mathrm{Span}(\{v_1, v_2\}) = \mathrm{Span}(S \setminus \{\vec{v}_3, \vec{v}_4\})$

• $\vec{v}_1 \parallel \vec{v}_2$?

If $\vec{v}_1 \parallel \vec{v}_2$ then $\begin{bmatrix} 3 \\ 3 \\ 5 \\ 2 \\ 4 \end{bmatrix} = k \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \\ 6 \end{bmatrix}$

$3 = 3k \to k = 1$
$3 = 2k \to k = 3/2$

impossible

***Theorem — The Minimizing Theorem:***

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors from $\mathbb{R}^m$, and let $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix}$ be the $m \times n$ matrix with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ as its ***columns***.

Suppose that $\boxed{R}$ is the rref of $A$, and $i_1, i_2, \ldots, i_k$ are the columns of $R$ that contain the ***leading variables***. Then the set $\boxed{S' = \{\vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_k}\}}$, that is, the subset of vectors of $S$ consisting of the corresponding columns of $A$, is a ***linearly independent*** set, and:

$$\boxed{Span(S) = Span(S').}$$

Furthermore, every $\vec{v}_i \in S - S'$, that is, the vectors of $S$ corresponding to the ***free variables*** of $R$, can be expressed as linear combinations of the vectors of $S'$, using the ***coefficients*** found in the corresponding column of $R$.

Pf Please read the book :)

☆ cols w/ leading 1s are LI

☆ Span(S) = Span(S')

potentially less vectors!

## *Idea:*

$$S = \left\{ \begin{array}{c} \overset{v_1}{\langle 7, 4, -3, 11 \rangle}, \overset{v_2}{\langle 2, -1, -1, 2 \rangle}, \overset{v_3}{\langle 31, 22, -13, 51 \rangle}, \\ \underset{v_4}{\langle 5, -2, 1, 5 \rangle}, \underset{v_5}{\langle 17, 12, -21, 29 \rangle} \end{array} \right\}.$$

$$A = \begin{array}{c} \begin{array}{ccccc} \text{LI} & \text{LI} & & \text{LI} & \end{array} \\ \begin{bmatrix} 7 & 2 & 31 & 5 & 17 \\ 4 & -1 & 22 & -2 & 12 \\ -3 & -1 & -13 & 1 & -21 \\ 11 & 2 & 51 & 5 & 29 \end{bmatrix} \end{array}$$

$$R = \begin{array}{c} \begin{bmatrix} 1 & 0 & 5 & 0 & 3 \\ 0 & 1 & -2 & 0 & 8 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & & \text{LD} & & \text{LD} \end{array} \end{array}.$$

for LD cols, use coefficients in the column to write LD equations.

Minimizing Thm $\Longrightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ is $\underline{\underline{\text{LI}}}$

& $\text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_4) = \text{Span}(S)$

$\uparrow$ long list

Magin:

$\vec{v}_3 = 5\vec{v}_1 - 2\vec{v}_2$

$$\begin{bmatrix} 31 \\ 22 \\ -13 \\ 51 \end{bmatrix} = 5 \begin{bmatrix} 7 \\ 4 \\ -3 \\ 11 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} ?$$

Surprisingly Deep          Useful

***Theorem — The Dependent vs. Spanning Sets Theorem:***

Suppose we have a set of $n$ vectors:

$$S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\},$$

from some Euclidean space $\mathbb{R}^k$, and we form $\boxed{Span(S)}$. Suppose now we randomly choose a set of $m$ vectors from $Span(S)$ to form a new set:

$$\boxed{L = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}}.$$

We can now conclude that if $\boxed{m > n}$, then $L$ is automatically linearly ***dependent***.

LD

In other words, if we chose ***more*** vectors from $Span(S)$ than the number of vectors we used to ***generate*** $S$, then this new set will certainly be ***dependent***.

***Proof:***

$$\vec{u}_1 = a_{1,1}\vec{w}_1 + a_{1,2}\vec{w}_2 + \cdots + a_{1,n}\vec{w}_n,$$

$$\vec{u}_2 = a_{2,1}\vec{w}_1 + a_{2,2}\vec{w}_2 + \cdots + a_{2,n}\vec{w}_n, \quad \ldots\ldots$$

$$\vec{u}_m = a_{m,1}\vec{w}_1 + a_{m,2}\vec{w}_2 + \cdots + a_{m,n}\vec{w}_n.$$

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_m\vec{u}_m = \vec{0}_k.$$

$$
\begin{aligned}
\vec{0}_k &= c_1(a_{1,1}\vec{w}_1 + a_{1,2}\vec{w}_2 + \cdots + a_{1,n}\vec{w}_n) + \\
&\quad c_2(a_{2,1}\vec{w}_1 + a_{2,2}\vec{w}_2 + \cdots + a_{2,n}\vec{w}_n) + \cdots + \\
&\quad c_n(a_{m,1}\vec{w}_1 + a_{m,2}\vec{w}_2 + \cdots + a_{m,n}\vec{w}_n) \\
&= c_1a_{1,1}\vec{w}_1 + c_1a_{1,2}\vec{w}_2 + \cdots + c_1a_{1,n}\vec{w}_n + \\
&\quad c_2a_{2,1}\vec{w}_1 + c_2a_{2,2}\vec{w}_2 + \cdots + c_2a_{2,n}\vec{w}_n + \cdots + \\
&\quad c_ma_{m,1}\vec{w}_1 + c_ma_{m,2}\vec{w}_2 + \cdots + c_ma_{m,n}\vec{w}_n \\
&= (c_1a_{1,1} + c_2a_{2,1} + \cdots + c_ma_{m,1})\vec{w}_1 + \\
&\quad (c_1a_{1,2} + c_2a_{2,2} + \cdots + c_ma_{m,2})\vec{w}_2 + \cdots + \\
&\quad (c_1a_{1,n} + c_2a_{2,n} + \cdots + c_ma_{m,n})\vec{w}_n.
\end{aligned}
$$

Now, we can ***force*** a solution if we set ***all*** of the coefficients of the vectors $\vec{w}_1$ through $\vec{w}_n$ to be zero:

$$c_1 a_{1,1} + c_2 a_{2,1} + \cdots + c_m a_{m,1} = 0,$$
$$c_1 a_{1,2} + c_2 a_{2,2} + \cdots + c_m a_{m,2} = 0, \ \ldots\ldots \ \text{and}$$
$$c_1 a_{1,n} + c_2 a_{2,n} + \cdots + c_m a_{m,n} = 0.$$

Contrapositive of "Dependent Sets from Span Thm"

is USEFUL!

***Theorem — The Independent vs. Spanning Sets Theorem:***

Suppose we have a set of $n$ vectors $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ from some Euclidean space $\mathbb{R}^k$, and we form $Span(S)$.

Suppose now we randomly choose a set of $m$ vectors from $Span(S)$ to form a new set:

$$L = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}.$$

We can now conclude that if $L$ is ***independent***, then $m \leq n$.

if $L$ is LI then $m \leq n$.

* Rmk actually the more important result (version).

Really beautiful proof & Theorem.

***Theorem — The Extension Theorem:***

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a ***linearly independent*** set of vectors from $\mathbb{R}^m$, and suppose $\vec{v}_{n+1}$ is ***not*** a member of $Span(S)$. Then, the extended set:

$$\boxed{S' = S \cup \{\vec{v}_{n+1}\}} \quad \text{"extend } S \text{ using } \vec{v}_{n+1}\text{"}$$
$$= \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{v}_{n+1}\}$$

is ***still linearly independent***.

- $\vec{v}_{n+1} \notin Span(\underbrace{\vec{v}_1, \ldots, \vec{v}_n}_{LI}) \Rightarrow \underbrace{\{\vec{v}_1, \ldots, \vec{v}_n, \vec{v}_{n+1}\}}_{S' =}$ LI

Pf Consider DTE:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n + c_{n+}\vec{v}_{n+1} = \vec{0}. \qquad (*)$$

Case $c_{n+1} = 0$

If $c_{n+1} = 0$ then $(*)$ simplifies to

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

Since $S$ is LI the only solution is if $c_1 = c_2 = c_3 = \cdots = c_n = 0$.

So $\vec{c} = \langle c_1, c_2, \ldots, c_{n+1}\rangle = \vec{0} \Rightarrow S'$ is LI.

Case $c_{n+1} \neq 0$

If $c_{n+1} \neq 0$ then $\vec{v}_{n+1} = \left(\frac{-c_1}{c_{n+1}}\right)\vec{v}_1 + \left(\frac{-c_2}{c_{n+1}}\right)\vec{v}_2 + \cdots + \left(\frac{-c_n}{c_{n+1}}\right)\vec{v}_n$

$\in Span(S)$ But we assumed $\vec{v}_{n+1} \notin Span(S)$.
This is a contradiction, Case 2 is impossible! $\square$