$\boxed{1.7}$

$S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\} \neq \emptyset.$

Thm $W = \text{Span}(S)$ is a subspace of $\mathbb{R}^n$.

Pf NTS 1) show W is not $\emptyset$
2) show W is closed under add
3) show W is closed under scal. mult.

(1) Recall $\vec{u}_1 \in S \subset \text{Span}(S) = W$

So $W \neq \emptyset$.

(2) let $\vec{w}_1, \vec{w}_2 \in W$. NTS $\vec{w}_1 + \vec{w}_2 \in W$.

$\vec{w}_1, \vec{w}_2 \in \text{Span}(S)$:

$$\vec{w}_1 = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k$$
$$\vec{w}_2 = d_1\vec{u}_1 + d_2\vec{u}_2 + \cdots + d_k\vec{u}_k$$

By properties of $\mathbb{R}^n$:

$$\vec{w}_1 + \vec{w}_2 = (c_1+d_1)\vec{u}_1 + \cdots + (c_k+d_k)\vec{u}_k$$

Thm 6, 7 of Prop. vec. Arith.

$\underbrace{\quad}_{\in \mathbb{R}}$ $\underbrace{\quad}_{\in \mathbb{R}}$ (A1 of $\mathbb{R}$)

so $\vec{w}_1 + \vec{w}_2$ is a LC of S.

$\vec{w}_1 + \vec{w}_2 \in \text{Span}(S) = W$.

(3) let $r \in \mathbb{R}$, $\vec{w} \in W$. NTS: $r\vec{w} \in W$.

exercise.

$\square$

Thm Dimension of Subspace Thm.

- B & B' are bases for W

Then $\boxed{\text{Card}(B) = \text{Card}(B')}$

ie both have same # of vectors in them.

Pf By def of basis: $\text{Span}(B) = W = \text{Span}(B')$.

Also $B \subseteq \text{Span}(B)$ so $B \subseteq \text{Span}(B')$

so B is LI (b/c it's a basis) & inside Span(B').

Thus, $\text{Card}(B) \leq \text{Card}(B')$, by the contrapositive to "Dependent set from Spanning set Thm":

- Similarly, $B' \subseteq Span(B') = Span(B)$,

So $B' \subseteq Span(B)$ & B' is LI

so by contrapositive to "DSFSST",

$$Card(B') \leq Card(B).$$

Thus, $Card(B) = Card(B')$. $\square$

<u>Thm</u> (Existence of a basis Thm)

<u>Pf Cont</u>. Use Mathematical Induction.

* <u>Base Case</u> already proved in Step 1.

* <u>IH</u>: Assume we have constructed a LI set

$$S_i = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i\} \text{ of } W$$

If $\mathrm{Span}(S_i) = W$, we're done

b/c S_i is a basis for W.

Otherwise, there exists $\vec{w}_{i+1} \in W \setminus \mathrm{Span}(S_i)$.

By Extension Theorem,

$S_{i+1} = S_i \cup \{\vec{w}_{n+1}\}$ is LI.

If $\mathrm{Span}(S_{i+1}) = W$, we're done.

Otherwise, can continue but this process

must stop. why? By the contrapositive

to the "Dependent Sets from Spanning Sets Thm."

(which says in R^n can have at most n LI vectors.)

So, by induction, can create a set

S_k for some $k \leq n$ that's a basis for W.

$\square$