# *1.7 Subspaces of Euclidean Spaces; Basis and Dimension*

***Definition:*** A ***subspace*** $W$ of $\mathbb{R}^n$ is a non-empty subset of vectors of $\mathbb{R}^n$ such that if $\vec{u}, \vec{v} \in W$, and $r \in \mathbb{R}$, then we also have:

$$\boxed{\vec{u} + \vec{v} \in W} \;\; \textbf{\textit{and}} \;\; \boxed{r \cdot \vec{v} \in W}.$$

We say $W$ is ***closed*** under vector addition and scalar multiplication, and write:

$$\boxed{W \trianglelefteq \mathbb{R}^n}$$

$A \subseteq B$ a subset $B$

$W \trianglelefteq \mathbb{R}^n$ subset $+$ closed

to indicate that $W$ is a subspace of $\mathbb{R}^n$. We call $\mathbb{R}^n$ the ***ambient space*** of $W$.

all subspaces contain $\vec{0}$

***Theorem:*** The zero vector $\vec{0}_n$ is always a member of any subspace $W \trianglelefteq \mathbb{R}^n$.

Pf let $W \trianglelefteq \mathbb{R}^n$. By def, $W \neq \emptyset$ so $\vec{w} \in W$.

zero Factor Thm: $0\vec{w} = \vec{0}$, W is close under scalar mult

$\vec{0} = 0\vec{w} \in W$. $\square$

***Definition/Theorem:*** For any $\mathbb{R}^n$, there are two ***trivial subspaces:*** (1) the subspace $\left\{\vec{0}_n\right\}$ consisting only of the zero vector, and (2) all of $\mathbb{R}^n$ itself.

$\{\vec{0}\} \trianglelefteq \mathbb{R}^n$ $\qquad$ $\mathbb{R}^n \trianglelefteq \mathbb{R}^n$

$\{\vec{0}\}$ & $\mathbb{R}^n$ trivial subspaces

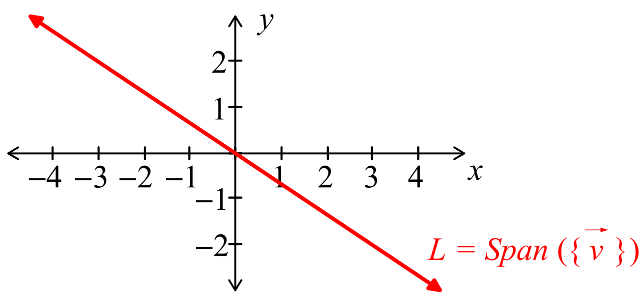
$\{\vec{0}\} \ntrianglelefteq W \ntrianglelefteq \mathbb{R}^n$

proper subspace.

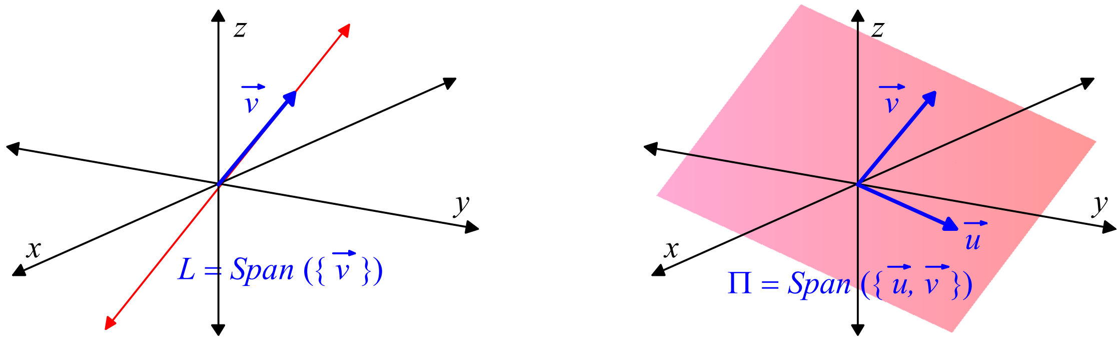
## *Span(S) as a Subspace*

***Theorem:*** If $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is a non-empty set of vectors from $\mathbb{R}^n$, then $\boxed{W = Span(S)}$ is a ***subspace*** of $\mathbb{R}^n$.

## *Examples:*




A Line Through the Origin is a Subspace of $\mathbb{R}^2$




# Lines or Planes Through the Origin are Subspaces of $\mathbb{R}^3$

## *Basis for a Subspace*

key concept

***Definition:*** A ***basis*** for a non-zero subspace $W \trianglelefteq \mathbb{R}^n$ is a non-empty set of vectors $\boxed{B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}}$ which ***Spans*** $W$ and is also ***linearly independent***.

Basis = LI + Span

***Example:*** "Standard basis" $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ for $\mathbb{R}^n$. □

LI ✓ Span ✓ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n$

***Example:***

line through the origin Span$(\{\vec{v}\})$, $\vec{v} \neq \vec{0}$.

↳ subspace

plane through the origin Span$(\{\vec{u}, \vec{v}\})$ $\vec{u} \nparallel \vec{v}$

↳ subspace.

# *A Basis for Span(S)*

***Theorem:*** The set $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a ***basis*** for $W = Span(B)$ ***if and only if*** $B$ is linearly independent.

$(\Rightarrow)$ B basis & $W = Span(B)$ $\Longrightarrow$ B is LI.

$(\Leftarrow)$ B is LI & $Span(B) = W$ $\Longrightarrow$ def of basis.

***Example:*** Suppose $S = \{\langle 1, 7, 3, -8, 2\rangle, \langle 4, -2, 5, 3, -4\rangle\}$.

LI or LD?

$\vec{v} = t\vec{w}$

$\vec{u} \parallel \vec{v}$

$\left.\begin{array}{l} 1 = 4t \\ 7 = -2t \end{array}\right\}$ contradiction!

not parallel! LI!

S forms a basis for $W = Span(\vec{v}, \vec{w})$.

subspace

USEFUL!!

***Theorem — The Minimizing Theorem (Basis for Span(S) Version):***

Suppose $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ and $W = Span(S)$. If $A = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{bmatrix}$, and $R$ is the rref of $A$, $\boxed{\text{then}}$ the columns of $A$ corresponding to the ***leading columns*** of $R$ form a ***basis*** for $W$.

MAGIC!!!

***Example:*** Suppose $\boxed{W = Span(S)}$, where:

$$S = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$$
$$= \{\langle 11, -13, -8, 17\rangle, \langle -4, 7, 3, -6\rangle, \langle 10, -5, -7, 16\rangle\}.$$

$$A = \begin{bmatrix} 11 & -4 & 10 \\ -13 & 7 & -5 \\ -8 & 3 & -7 \\ 17 & -6 & 16 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{array}{ccc} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \text{LI} \quad \text{LI} \quad \text{LD} \end{array}$$

* note we haven't done this before!

A is not a SOE's! Purely the info of S.

col 3 = 2 col 1 + 3 col 2

$$\begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

* $\{\vec{w}_1, \vec{w}_2\}$ is LI

* $Span(\vec{w}_1, \vec{w}_2) = Span(S) = W$

SO $\{\vec{w}_1, \vec{w}_2\}$ is a basis for $W$.

## *Constructing a Basis for Any Subspace*

IMPORTANT RESULT DEEP

***Theorem — Existence of a Basis Theorem:***

If $W$ is any ***non-zero*** subspace of $\mathbb{R}^n$, then ***there exists*** a basis $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ for $W$. In other words, we can write:

$$W = Span(B) = Span(\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}),$$

where $B$ is a linearly independent set that Spans $W$. Furthermore, we must have $k \leq n$.

Pf Assume $W \neq \emptyset$, pick some $\vec{w}_1 \in W$. Can assume $\vec{w} \neq \vec{0}$.

Step1 $S_1 = \{\vec{w}_1\}$. Since $\vec{w}_1 \neq \vec{0}$ $\{\vec{w}_1\}$ is LI.

So if $Span(S_1) = W$, we're done. (since $S_1$ is a basis for $W$)

Step2 Otherwise, there exist $\vec{w}_2 \in W \setminus Span(S_1)$

In particular, $\vec{w}_2 \neq \vec{0}$.

$S_2 = \{\vec{w}_1, \vec{w}_2\}$.

So by the Extension Theorem (§1.6), $S_2$ is LI.

So if $Span(S_2) = W$, we're done (since $S_2$ is a basis)

In general, we can continue in this manner.

*Theorem — The Subspaces of Euclidean 2-Space and 3-Space:*

The only subspaces of $\mathbb{R}^2$ are:

(a) the zero subspace $\left\{\vec{0}_2\right\}$,

(b) the ***lines*** through the origin, and

(c) all of $\mathbb{R}^2$.

Similarly, the only subspaces of $\mathbb{R}^3$ are:

(a) the zero subspace $\left\{\vec{0}_3\right\}$,

(b) the ***lines*** through the origin,

(c) the ***planes*** through the origin, and

(d) all of $\mathbb{R}^3$.

$\mathbb{R}$

- $\{0\}$
- ?
- $\mathbb{R}$

no proper subspaces!

$\mathbb{R}^2$

- $\{\vec{0} = \langle 0, 0\rangle\}$
- $W$ = lines through origin
- $\mathbb{R}^2$

$\mathbb{R}^3$

- $\vec{0}$
- lines thru origin
- planes thru origin
- $\mathbb{R}^3$

## *The Dimension of a Subspace*

the actual "theorem" part

**Theorem/Definition — The Dimension of a Subspace:**

If $B$ and $B'$ are any two bases for the same non-zero subspace $W \trianglelefteq \mathbb{R}^n$, then $B$ and $B'$ contain ***exactly the same number of vectors***. We call this number the ***dimension*** of $W$, and we write $\boldsymbol{dim}(W) = k$. We also say that $W$ is $k$***-dimensional***.

We agreed that the trivial subspace $\left\{\vec{0}_n\right\}$ does ***not*** have a basis. By convention, $dim\left(\left\{\vec{0}_n\right\}\right) = 0$.

Conversely, $dim(W)$ is a ***positive integer*** for a ***non-zero*** subspace $W$.

$B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ basis of $W$

dimension.

Notation $\#B = \|B\| = Card(B) = \#$ of vectors in $B$

$\{\vec{0}\}$, basis for this? $B = \emptyset$. $Card(\emptyset) = 0$.

technicality

$\{\vec{0}\}$ is LD $\to$ can't be a basis.

*Example:* $dim(\mathbb{R}^n) = ?$

*Example:*

line through the origin

plane through the origin

***Example:*** $W = Span(S)$, where:

$$S = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$$
$$= \left\{ \begin{array}{c} \langle 11, -13, -8, 17\rangle, \langle -4, 7, 3, -6\rangle, \\ \langle 10, -5, -7, 16\rangle \end{array} \right\}.$$

ambient space? $\mathbb{R}^4$

maximum possible dimension? 3

basis? $\mathcal{B}$

actual dimension? 2

test if S is LI or LD:

$$\begin{bmatrix} 11 & -4 & \cancel{10} \\ -13 & 7 & \cancel{-5} \\ -8 & 3 & \cancel{-7} \\ 17 & -6 & \cancel{16} \end{bmatrix} \xrightarrow{RREF} \begin{array}{ccc} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \uparrow \quad \uparrow \quad \uparrow \text{ LD} \\ \underbrace{\text{LI} \quad \text{LI}}_{\text{\& Spanning original}} \end{array}$$

Basis

$$B = \left\{ \begin{bmatrix} 11 \\ -13 \\ -8 \\ 17 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \\ 3 \\ -6 \end{bmatrix} \right\} \text{ is a basis}$$

*Example:* $W = Span(S)$, where:

$$S = \{\langle 3,-2,5,4,1,-6\rangle, \langle 5,-3,6,3,2,-8\rangle,$$
$$\langle -11,5,-2,11,-6,8\rangle,$$
$$\langle -2,1,-4,-2,0,6\rangle, \langle -4,1,-1,7,-1,6\rangle\}.$$

ambient space?

maximum possible dimension?

$$A = \begin{bmatrix} 3 & 5 & -11 & -2 & -4 \\ -2 & -3 & 5 & 1 & 1 \\ 5 & 6 & -2 & -4 & -1 \\ 4 & 3 & 11 & -2 & 7 \\ 1 & 2 & -6 & 0 & -1 \\ -6 & -8 & 8 & 6 & 6 \end{bmatrix}, \quad \text{with rref:}$$

$$R = \begin{bmatrix} 1 & 0 & 8 & 0 & 5 \\ 0 & 1 & -7 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$