1.8

transpose of A

$A^T = (a_{ji})$ = turn rows of A into cols of A^T

j^{th} column of A

$\vec{x} \in \mathbb{R}^n$

i^{th} row of A

$$A = (a_{ij})_{m \times n} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cc} \left[\begin{array}{cccccc} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{array}\right] & \begin{array}{c} \vec{r}_1 \\ \vec{r}_2 \\ \\ \vec{r}_i \\ \\ \vec{r}_m \end{array} \\ \begin{array}{cccccc} \vec{c}_1 & \vec{c}_2 & & \vec{c}_j & & \vec{c}_n \end{array} & \end{array} \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix}$$

$\vec{c} \in \mathbb{R}^m$ $\qquad$ $\vec{r} \in \mathbb{R}^n$

★ **FSSM1** RowSpace(A) = RS(A) = $\text{Span}(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m) \leq \mathbb{R}^n$

★ **FSSM2** Column Space(A) = CS(A) = $\text{Span}(\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n) \leq \mathbb{R}^m$

★ **FSSM3** Null Space(A) = NS(A)

$$= \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}_m \}$$

★ **FSSM4** Null Space(A^T) = NS(A^T)

$$= \{ \vec{y} \in \mathbb{R}^m \mid A^T \vec{y} = \vec{0}_n \}$$

transpose of A $= A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{i1} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{i2} & \cdots & a_{m2} \\ & & & \vdots & & & \\ a_{1j} & a_{2j} & a_{3j} & \cdots & a_{ij} & \cdots & a_{mj} \\ & & & \vdots & & & \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{in} & \cdots & a_{mn} \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_m \end{bmatrix}$

$n \times m$

Ex $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

$A_{2\times 3}$ $\qquad A^T = 3\times 2$

Fun Fact $A \cdot A^T$ & $A^T \cdot A$ are always defined.

$\underbrace{m\times \underbrace{n \quad n}_{=} \times m}_{m\times m} \qquad \underbrace{n\times \underbrace{m \quad m}_{=} \times n}_{n\times n}$

Ex • RS(A) 4 rows in $\mathbb{R}^7$ these could be LI.

• non-zero rows of R form a basis for RS(A).
(LI + Span)

• $RS(A) = \text{Span}\left(\left\{ \langle 1, -4, 0, 3, 0, 5, 6 \rangle, \langle 0, 0, 1, -2, 0, 7, -3 \rangle, \langle 0, 0, 0, 0, 1, -8, 4 \rangle \right\}\right)$

• $\dim(RS(A)) = 3$.

• $RS(A) \trianglelefteq \mathbb{R}^7$.

• Note worth checking these are indeed LI.

Magic can show original rows are LD (ie LCs of these three vectors)

ex

$$\begin{bmatrix} 7 \\ -28 \\ 2 \\ 11 \\ -3 \\ 25 \\ 24 \end{bmatrix} = 7\begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \\ 0 \\ 5 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 0 \\ 7 \\ -3 \end{bmatrix} -3\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -8 \\ 4 \end{bmatrix}$$

Can do this for all rows in original A.

$$RS(A) \subseteq RS(R).$$

• CS(A) there's 7 vectors from $\mathbb{R}^4$

so the cols of A are LD.

- Thm says: the corresponding $\overset{\text{col}}{\vee}$ vectors in A to the columns in R w/ leading 1s these form a basis for $CS(A)$.

- $CS(A) = \text{Span}\left(\left\{ \begin{bmatrix} 7 \\ -3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 24 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 4 \\ 4 \end{bmatrix} \right\}\right)$
  $\qquad\qquad\qquad\qquad\qquad \vec{c}_1 \qquad\quad \vec{c}_3 \qquad\quad \vec{c}_5$
- $\dim(CS(A)) = 3$

- Rest of cols are LD:

In R col w/ free:

$$\underbrace{\begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{col } 2} = -4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{col } 1} \xrightarrow{\text{magic}} \begin{bmatrix} -28 \\ 12 \\ 4 \\ -8 \end{bmatrix} = -4 \begin{bmatrix} 7 \\ -3 \\ -1 \\ 2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix}}_{\text{col } 4} = 3 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{col } 1} - 2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{col } 3} \xrightarrow{\text{magic}} \begin{bmatrix} 17 \\ -17 \\ -51 \\ 12 \end{bmatrix} = 3 \begin{bmatrix} 7 \\ -3 \\ -1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 4 \\ 24 \\ -3 \end{bmatrix}$$

$$\boxed{\vec{c}_4 = 3\vec{c}_1 - 2\vec{c}_3}$$

etc.... $\vec{c}_6, \vec{c}_7 \in \text{Span}(\vec{c}_1, \vec{c}_3, \vec{c}_5)$

- <u>$NS(A)$</u> $\quad A\vec{x} = \vec{0} \longleftrightarrow R\vec{x} = \vec{0}$

- $\boxed{x_1} - 4x_2 + 3x_4 + 5x_6 + 6x_7 = 0$

  $\boxed{x_3} - 2x_4 + 7x_6 - 3x_7 = 0$

$$\boxed{x_5} - 8x_6 + 4x_7 = 0$$

$$0 = 0$$

- note: leading 1 variables & free variables:

leading 1 : x_1, x_3, x_5

free : x_2, x_4, x_6, x_7

EQs
$$\begin{cases} x_1 = 4x_2 - 3x_4 - 5x_6 - 6x_7 \\ x_3 = 2x_4 - 7x_6 + 3x_7 \\ x_5 = 8x_6 - 4x_7 \end{cases}$$

Recall $NS(A) : \{\vec{x} \mid A\vec{x} = \vec{0}\}. =$

$$\left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 4x_2 - 3x_4 - 5x_6 - 6x_7 \\ x_2 \\ 2x_4 - 7x_6 + 3x_7 \\ x_4 \\ 8x_6 - 4x_7 \\ x_6 \\ x_7 \end{bmatrix} : x_2, x_4, x_6, x_7 \in \mathbb{R} \right\}$$

$$\vec{x} = x_2 \underbrace{\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{z}_1} + x_4 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{z}_2} + x_6 \underbrace{\begin{bmatrix} -5 \\ 0 \\ -7 \\ 0 \\ 8 \\ 1 \\ 0 \end{bmatrix}}_{\vec{z}_3} + x_7 \underbrace{\begin{bmatrix} -6 \\ 0 \\ 3 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}}_{\vec{z}_4}$$

$$NS(A) = \text{Span}(\{\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4\}).$$

Note $\{\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4\}$ is LI & basis.

- $NS(A^T)$: $\{ \vec{y} \in \mathbb{R} \mid A^T \vec{y} = \vec{0} \}$

$$A^T = \begin{bmatrix} 7 & -3 & -1 & 2 \\ -28 & 12 & 4 & -8 \\ 2 & 4 & 24 & -3 \\ 17 & -17 & -51 & 12 \\ -3 & 3 & 4 & 4 \\ 23 & -3 & 131 & -43 \\ 24 & -22 & -62 & 37 \end{bmatrix} \xrightarrow{RREF} \underbrace{\begin{array}{cccc} y_1 & y_2 & \boxed{y_3} & y_4 \\ \hline 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}}_{7\times 4} \ \underbrace{\overset{\vec{y}}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}}_{4\times 1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{7\times 1}$$

EQ
$$\left\{ \begin{array}{l} y_1 + 2y_3 = 0 \to y_1 = -2y_3 \\ y_2 + 5y_3 = 0 \to y_2 = -5y_3 \\ y_4 = 0 \to y_4 = 0 \end{array} \right\} \Rightarrow \vec{y} = \begin{bmatrix} -2y_3 \\ -5y_3 \\ y_3 \\ 0 \end{bmatrix}$$

leading 1s var: y_1, y_2, y_4

free : y_3

$$\vec{y} = y_3 \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \end{bmatrix}.$$

$$NS(A^T) = \text{Span}\left(\left\{ \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \end{bmatrix} \right\} \right) \trianglelefteq \mathbb{R}^4$$

$$\dim(NS(A^T)) = 1$$