$\boxed{1.9}$

<u>Thm</u> $W^\perp \trianglelefteq \mathbb{R}^n$.

<u>Pf</u> • $W^\perp \neq \emptyset$

$W^\perp = \{ \vec{v} \in \mathbb{R}^n \mid \boxed{\vec{v} \circ \vec{w} = 0}, \forall \vec{w} \in W \}$.

Observe: $\vec{0} \circ \vec{w} = 0 \ \forall \vec{w} \in W$

so $\vec{0} \in W^\perp$.

• <u>closed under +</u>

let $\vec{v}_1, \vec{v}_2 \in W^\perp$. <u>NTS</u> $\vec{v}_1 + \vec{v}_2 \in W^\perp$.

Then def. of $W^\perp$ says:

$$\vec{v}_1 \circ \vec{w} = 0 \ \forall \vec{w} \in W.$$

$$\& \ \vec{v}_2 \circ \vec{w} = 0 \ \forall \vec{w} \in W.$$

Let $\vec{w} \in W$ be arbitrary.

Then

$$(\vec{v}_1+\vec{v}_2)\circ\vec{w} = \vec{v}_1\circ\vec{w} + \vec{v}_2\circ\vec{w}$$

$$= 0 + 0$$

$$= 0.$$

So, $\vec{v}_1+\vec{v}_2 \in W^\perp$.

- Scal Mult

Sketch $k\vec{v} \in W^\perp$?

$$(k\vec{v})\circ\vec{w} = k(\vec{v}\circ\vec{w})$$

$$= k\cdot 0$$

$$= 0 \qquad \square$$

Thm Hint $\vec{v} \in W^\perp$ $\vec{w} \in W = \text{Span}(\vec{w}_1, \dots, \vec{w}_k)$

$$\vec{w} = x_1\vec{w}_1 + x_2\vec{w}_2 + \dots + x_k\vec{w}_k.$$

Then $\vec{v}\circ\vec{w} = 0 \Rightarrow \dots$

EX

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 5 & 0 \\ -2 & 5 & 7 & -8 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & * & * & 0 \\ 0 & 1 & * & * & 0 \end{array}\right)$$

$$2R_1 + R_2 \to R_2 \qquad \longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & -2 & 5 & 0 \\ 0 & 11 & 3 & 2 & 0 \end{array}\right)$$

$$\frac{1}{11}R_2 \to R_2 \qquad \longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & -2 & 5 & 0 \\ 0 & 1 & \frac{3}{11} & \frac{2}{11} & 0 \end{array}\right)$$

$$-3R_2 + R_1 \to R_1 \qquad \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & \overset{-31/11}{\boxed{-9/11 - 2}} & \overset{49/11}{\boxed{-\frac{6}{11} + 5}} & 0 \\ 0 & 1 & \frac{3}{11} & \frac{2}{11} & 0 \end{array}\right)$$

solutions to this HSOE iff NS(A)

$$\begin{cases} x_1 - \frac{31}{11}x_3 + \frac{49}{11}x_4 = 0 \\ x_2 + \frac{3}{11}x_3 + \frac{2}{11}x_4 = 0 \end{cases}$$

$$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{31}{11}x_3 - \frac{49}{11}x_4 \\ -\frac{3}{11}x_3 - \frac{2}{11}x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

leading: x_1, x_2

free: x_3, x_4

$$\vec{v} = x_3 \begin{bmatrix} 31/11 \\ -3/11 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -49/11 \\ -2/11 \\ 0 \\ 1 \end{bmatrix}$$

$$W^\perp = \operatorname{Span}\left(\left\{ \begin{bmatrix} 31/11 \\ -3/11 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -49/11 \\ -2/11 \\ 0 \\ 1 \end{bmatrix} \right\}\right)$$

$$= \operatorname{Span}\left(\left\{ \begin{bmatrix} 31 \\ -3 \\ 11 \\ 0 \end{bmatrix}, \begin{bmatrix} -49 \\ -2 \\ 0 \\ 11 \end{bmatrix} \right\}\right)$$

can multiply by non-zero scalars

Thm (2 for 1 Theorem) (useful)

Assume $W \leq \mathbb{R}^n$, $\dim(W) = k$

Let $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$.

B is a $\boxed{\text{basis}}$ $\boxed{\text{iff}}$ B is LI or $\text{Span}(B) = W$.

If ($\Longrightarrow$) Assume B is a basis.

Then B is LI & $\text{Span}(B) = W$

are both true! Done.

($\Longleftarrow$) Assume B is LI or $\text{Span}(B) = W$.

Case 1 B is LI

Argue by contradiction. Assume that B not a basis. Then B is not spanning, so there's $\vec{w} \in W \setminus \text{Span}(B)$. By Extension Theorem,

$B' = \underset{\text{LI}}{\underset{\uparrow}{B}} \cup \{\vec{w}\}$ is also LI. If B' is a basis for W then since B' is LI we get $\dim(W) = k+1$. But $\dim(W) = k$ by assumption which is a contradiction! Why? Dependent Set from Spanning Set Theorem says if $\dim(W) = k$ can have at most k LI vectors.

Thus, B must be a basis.

<u>Case 2</u> $\text{Span}(B) = W$.

<u>WTS</u> B is a basis, need only show B is LI.

Argue by contradiction. Assume B is not LI.

By Minimizing Theorem, there exist a subset $B' \subset B$ so that B' is LI and $\text{Span}(B') = W$.

This means B' is a basis! If $\text{card}(B') = k$ then $B' = B$, so B is a basis. Otherwise,

$\text{Card}(B') < K$. Then since B' is a basis for W, $\dim(W) = \text{Card}(B') < K$.

This contradicts our assumption. So, B is LI. So B is a basis.

$\square$