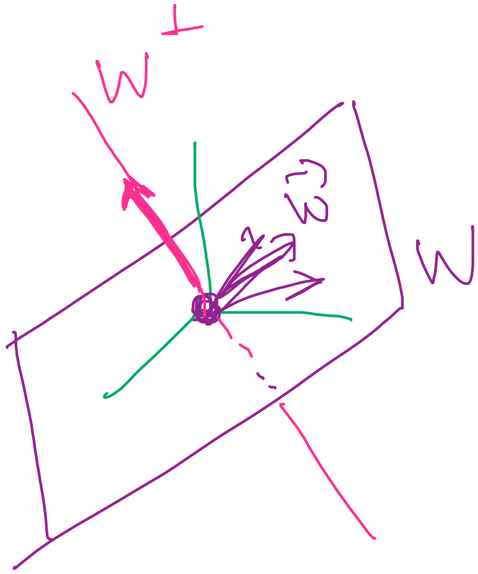
Section 1.9 Orthogonal Complements

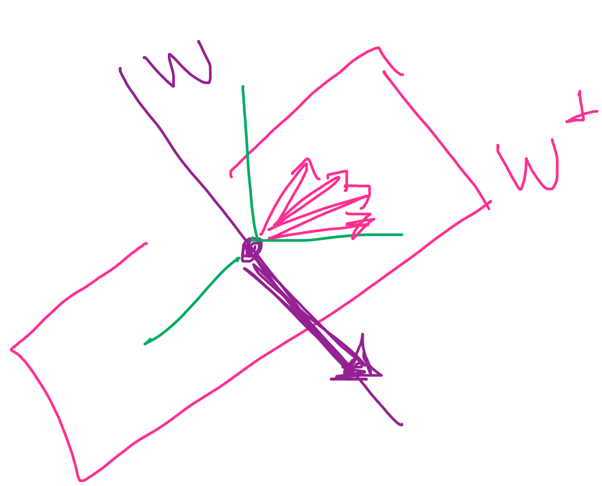
☆ Uses dot product → gives 0

Definition/Theorem: If W is a subspace of $\mathbb{R}^n$, then $W^\perp$ (pronounced ***"W perp"***), the ***orthogonal complement*** of W, defined as:

$$W^\perp = \left\{ \vec{v} \in \mathbb{R}^n \mid \boxed{\vec{v} \circ \vec{w} = 0} \text{ for } \textbf{\textit{all}}\ \vec{w} \in W \right\}$$

is also a ***subspace*** of $\mathbb{R}^n$.





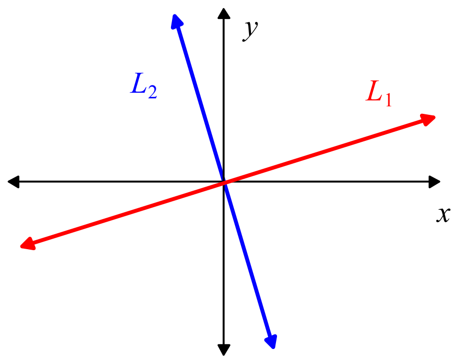
Thm

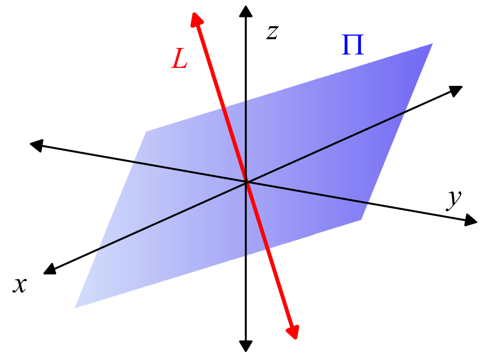
$W^\perp \trianglelefteq \mathbb{R}^n$.

$W^\perp \trianglelefteq \mathbb{R}^n$

$W \trianglelefteq \mathbb{R}^n$

Examples:





$$L_1 : y = mx,\ m \neq 0, \text{ and } L_2 : y = -\frac{1}{m}x$$

$$\Pi : ax + by + cz = 0, \text{ and } L = Span(\{\langle a, b, c \rangle\})$$

Theorem: If $W = Span(\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}) \trianglelefteq \mathbb{R}^n$, then:

$$W^{\perp} = \left\{\vec{v} \in \mathbb{R}^n \,|\, \underline{\vec{v} \circ \vec{w}_i = 0} \text{ for all } i = 1\ldots k\right\}.$$

ie check $\vec{v}$ is $\perp$ to just the k vectors: $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$

Example: Suppose $S = \{\langle \overset{\vec{w}_1}{1}, 3, -2, 5\rangle, \langle \overset{\vec{w}_2}{-2}, 5, 7, -8\rangle\}$ and $W = Span(S) \trianglelefteq \mathbb{R}^4$.

$$\begin{bmatrix} 1 \\ 3 \\ -2 \\ 5 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 5 \\ 7 \\ -8 \end{bmatrix}$$

Let us find a basis for $W^{\perp}$.

Set up $\vec{v} \in W^{\perp}$ iff $\begin{cases} \vec{v} \circ \vec{w}_1 = 0 & (*) \\ \vec{v} \circ \vec{w}_2 = 0 & (**) \end{cases}$

$\vec{v} \in \mathbb{R}^4 \quad \vec{v} = \langle x_1, x_2, x_3, x_4 \rangle$

$$\begin{cases} \vec{v} \circ \vec{w}_1 = \langle x_1, x_2, x_3, x_4 \rangle \circ \langle 1, 3, -2, 5 \rangle = 0 \\ v \circ \vec{w}_2 = \langle \quad " \quad \rangle \circ \langle -2, 5, 7, -8 \rangle = 0 \end{cases}$$

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 5x_4 = 0 \\ -2x_1 + 5x_2 + 7x_3 - 8x_4 = 0 \end{cases} \quad \left(\boxed{\begin{matrix} 1 & 3 & -2 & 5 \\ -2 & 5 & 7 & -8 \end{matrix}} \;\middle|\; \begin{matrix} 0 \\ 0 \end{matrix} \right)$$

Note $W^{\perp} = NS(A)$ where A is given by

rows of A = dot product w/ $\vec{v} \in W^{\perp}$

A Dot Product Perspective of Matrix Multiplication

$A \vec{x} = \vec{b}_{m \times 1}$ with A: $m \times n$, $\vec{x}$: $n \times 1$

dot product!

"row picture" SOE

$$A\vec{x} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots \\ a_{21}x_1 + a_{22}x_2 + \cdots \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots \end{bmatrix}$$

column picture LCs & Spans

$$= x_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix} + x_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m,2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{bmatrix}$$

$$x_1 a_{1,1} + x_2 a_{1,2} + \cdots + x_n a_{1,n}$$
$$= \langle x_1, x_2, \ldots, x_n \rangle \circ \langle a_{1,1}, a_{1,2}, \ldots a_{1,n} \rangle$$
$$= \vec{x} \circ \vec{r}_1,$$

$$A\vec{x} = \vec{b} = \begin{bmatrix} \vec{r}_1 \circ \vec{x} \\ \vec{r}_2 \circ x \\ \vdots \\ \vec{r}_m \circ \vec{X} \end{bmatrix}_{m \times 1}$$

$$A\vec{x} = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vdots \\ \vec{r}_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \vec{x} \circ \vec{r}_1 \\ \vec{x} \circ \vec{r}_2 \\ \vdots \\ \vdots \\ \vec{x} \circ \vec{r}_m \end{bmatrix} = \begin{bmatrix} \vec{r}_1 \circ \vec{x} \\ \vec{r}_2 \circ \vec{x} \\ \vdots \\ \vdots \\ \vec{r}_m \circ \vec{x} \end{bmatrix}.$$

$$A = \begin{bmatrix} | & | & & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & & | \end{bmatrix}$$

Thm $\vec{x} \in \mathbb{R}^n$, solution to $A\vec{x} = \vec{0}$ (HSDE)

$\boxed{\text{iff}}$ $\vec{r}_i \circ \vec{x} = 0$ for all rows of A, $i = 1, 2, \ldots, m$

$\boxed{\text{iff}}$ $\vec{x} \in NS(A)$

$\boxed{\text{iff}}$ $\vec{x} \in RS(A)^{\perp}$

Theorem: A vector $\vec{x} \in \mathbb{R}^n$ is a solution to $A\vec{x} = \vec{\mathbf{0}}_m$ ***if and only if*** $\vec{x} \circ \vec{r}_i = 0$ for all the rows $\vec{r}_i$ of A. In other words, $\vec{x}$ is in the ***nullspace*** of A ***if and only if*** $\vec{x}$ is ***orthogonal*** to all the ***rows*** of A. Thus:

$$\text{If } \boxed{W = rowspace(A),} \text{ then } \boxed{W^{\perp} = nullspace(A).}$$

$$\text{Similarly, if } \boxed{U = nullspace(A),} \text{ then } \boxed{U^{\perp} = rowspace(A).}$$

Warning put into rows! This is only section we do this

Theorem: Suppose $W = Span(\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}) \trianglelefteq \mathbb{R}^n$. If we form the matrix A with ***rows*** $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$, then:

$$\boxed{W = rowspace(A)} \quad \text{and} \quad \boxed{W^\perp = nullspace(A)}.$$

Thus, the non-zero rows of the rref of A form a basis for W, and we can obtain a basis for $W^\perp$ exactly as we would find a basis for $nullspace(A)$ using the rref of A.

Note: This is the ***only*** place in this book where we assemble vectors into the ***rows*** of a matrix. The rest of the time, we will assemble vectors into the columns of a matrix.

Example: $W = Span(S) \trianglelefteq \mathbb{R}^5$, where:

$$S = \{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$$
$$= \left\{ \begin{array}{c} \langle 3, -2, 1, 5, 0\rangle, \langle 5, -3, 2, 6, 1\rangle, \\ \langle -8, 3, -5, 3, -7\rangle, \langle 4, 1, 0, -2, 3\rangle \end{array} \right\}.$$

Find $W^\perp$

Use rows

$$A = \begin{bmatrix} 3 & -2 & 1 & 5 & 0 \\ 5 & -3 & 2 & 6 & 1 \\ -8 & 3 & -5 & 3 & -7 \\ 4 & 1 & 0 & -2 & 3 \end{bmatrix}$$

$$\xrightarrow{RREF} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 2/5 & 2/5 \\ 0 & 1 & 0 & -18/5 & 7/5 \\ 0 & 0 & 1 & -17/5 & 8/5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

leading : x_1, x_2, x_3

free : x_4, x_5

$$\boxed{W^\perp} = NS(A) = Span\left(\left\{ \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \right\}\right)$$

$$= Span\left(\left\{ \begin{bmatrix} -2/5 \\ 18/5 \\ 17/5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2/5 \\ -7/5 \\ -8/5 \\ 0 \\ 1 \end{bmatrix} \right\}\right).$$

$$\boxed{W} = (NS(A))^\perp = RS(A) = Span\left(\left\{ \begin{array}{c} \langle 1, 0, 0, 2/5, 2/5 \rangle \\ \langle 0, 1, 0, -18/5, 7/5 \rangle \\ \langle 0, 0, 1, -17/5, 8/5 \rangle \end{array} \right\}\right).$$

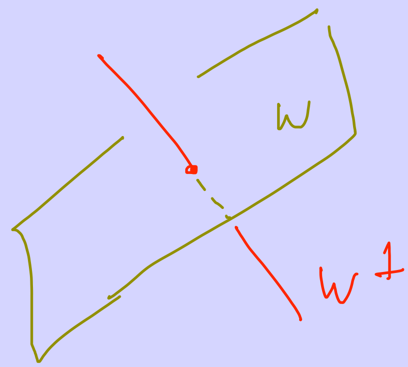
$\dim(W) + \dim(W^{\perp}) = 3 + 2 = 5.$

Theorem — Properties of Orthogonal Complements:

For any subspace $W \trianglelefteq \mathbb{R}^n$:

a) $W \cap W^{\perp} = \left\{\vec{0}_n\right\}$

b) $(W^{\perp})^{\perp} = W$.



Thus, we can say that W and $W^{\perp}$ are orthogonal complements of ***each other***, or that W and $W^{\perp}$ form an ***orthogonal pair*** of subspaces.

Pf Excercise. $\square$

Theorem — The Dimension Theorem for Orthogonal Complements:

If W is a subspace of $\mathbb{R}^n$ with orthogonal complement $W^\perp$, then: $dim(W) + dim(W^\perp) = n$.

Pf. By dim Thm: $\dim(RS(A)) + \dim(NS(A)) = n$ where A is obtained by putting the basis of W into rows (since W has a basis by our thm). □.

Example: Suppose that:

$$S = \{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$$

$$= \left\{ \begin{array}{c} \langle 15, -10, 5, 25, 0\rangle, \langle -9, 6, -3, -15, 0\rangle, \\ \langle 1, -2, -5, 11, 8\rangle, \langle 5, -3, 3, 6, -2\rangle \end{array} \right\}$$

$$\subset \mathbb{R}^5,$$

~~and~~ $W = Span(S)$. Find W & $W^\perp$.

use rows

$$A = \begin{bmatrix} 15 & -10 & 5 & 25 & 0 \\ -9 & 6 & -3 & -15 & 0 \\ 1 & -2 & -5 & 11 & 8 \\ 5 & -3 & 3 & 6 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{array}{c} \begin{array}{ccccc} & & \text{free} & \text{free} & \text{free} \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 3 & -3 & -4 \\ 0 & 1 & 4 & -7 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

- $W = RS(A) = RS(R) = Span(\{\langle 1, 0, 3, -3, -4\rangle, \langle 0, 1, 4, -7, -6\rangle\})$.
- $W^\perp = NS(A) = NS(R) = Span\left(\left\{ \begin{bmatrix} -3 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}\right)$

Using dim(W) to Find Other Bases for W

Theorem — The "Two for the Price of One" or "Two-for-One" Theorem:

Suppose W is a subspace of $\mathbb{R}^n$, and $dim(W) = k$. Let $B = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_k\}$ be any subset of k vectors from W. Then: B is a basis for W ***if and only if*** either B is linearly independent ***or*** B Spans W. In other words, it is necessary and sufficient to check B for ***only one condition*** without checking the other, if B already contains the ***correct number of vectors***.

easier to ***check linear independence*** rather than Spanning,

Example: In the previous Example:

$$S = \{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$$

$$= \left\{ \begin{array}{c} \langle 15, -10, 5, 25, 0\rangle, \langle -9, 6, -3, -15, 0\rangle, \\ \langle 1, -2, -5, 11, 8\rangle, \langle 5, -3, 3, 6, -2\rangle \end{array} \right\}.$$

We found that $\dim(W) = 2,$ and a basis for W is the set:

$$B = \{\langle 1, 0, 3, -3, -4\rangle, \langle 0, 1, 4, -7, -6\rangle\}.$$

Example: Suppose we have $W = Span(S) \trianglelefteq \mathbb{R}^5$, where:

$$S = \{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4, \vec{w}_5\}$$

$$= \left\{ \begin{array}{c} \langle 3,-2,4,1,-3\rangle, \langle 1,-1,2,-1,0\rangle, \\ \langle 7,-4,8,5,-9\rangle, \\ \langle 1,2,-1,0,3\rangle, \langle 5,-1,2,11,-12\rangle \end{array} \right\}.$$

$$A = \begin{bmatrix} 3 & -2 & 4 & 1 & -3 \\ 1 & -1 & 2 & -1 & 0 \\ 7 & -4 & 8 & 5 & -9 \\ 1 & 2 & -1 & 0 & 3 \\ 5 & -1 & 2 & 11 & -12 \end{bmatrix},$$

with rref:

$$R = \begin{bmatrix} 1 & 0 & 0 & 3 & -3 \\ 0 & 1 & 0 & -\frac{10}{3} & 5 \\ 0 & 0 & 1 & -\frac{11}{3} & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$A^\top = \begin{bmatrix} 3 & 1 & 7 & 1 & 5 \\ -2 & -1 & -4 & 2 & -1 \\ 4 & 2 & 8 & -1 & 2 \\ 1 & -1 & 5 & 0 & 11 \\ -3 & 0 & -9 & 3 & -12 \end{bmatrix}$$

with rref:

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$