2.1

"Equivalence of LT & Mat Thm"

Thm $T : \mathbb{R}^n \to \mathbb{R}^m$

T is a Linear Transformation **iff** $\exists$ matrix $A_{m\times n}$ so that

$$\boxed{T(\vec{x}) = A\vec{x}}$$

Pf ($\Rightarrow$) Recall: $B = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$

Assume T is a LT.

Let $\vec{x} \in \mathbb{R}^n$. Then

$$\vec{x} = \langle x_1, x_2, x_3, \ldots, x_n\rangle$$

$$= x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n \quad (LC)$$

Then

$$T(\vec{x}) = T(x_1\vec{e}_1 + (x_2\vec{e}_2 + \cdots + x_n\vec{e}_n))$$

$$= T(x_1\vec{e}_1) + T(x_2\vec{e}_2 + \cdots + x_n\vec{e}_n)$$

(by def of T: additivity prop)

$= \cdots\cdots$

$= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \cdots + T(x_n\vec{e}_n)$

$$\boxed{T(\vec{x}) = x_1 \underbrace{T(\vec{e}_1)}_{\text{vectors in } \mathbb{R}^m} + x_2 \underbrace{T(\vec{e}_2)} + \cdots + x_n \underbrace{T(\vec{e}_n)}}$$

(by Homogenity Prop)

Thus, for scalars $a_{ij} \in \mathbb{R}$: for $j = 1, 2, \ldots, n$.

$$T(\vec{e}_j) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = \langle a_{1j}, a_{2j}, \ldots, a_{mj} \rangle \in \mathbb{R}^m$$

So:

$$T(\vec{x}) = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= \overbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix}_{m\times n}}^{A} \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}^{\vec{x}} = A\vec{x}$$

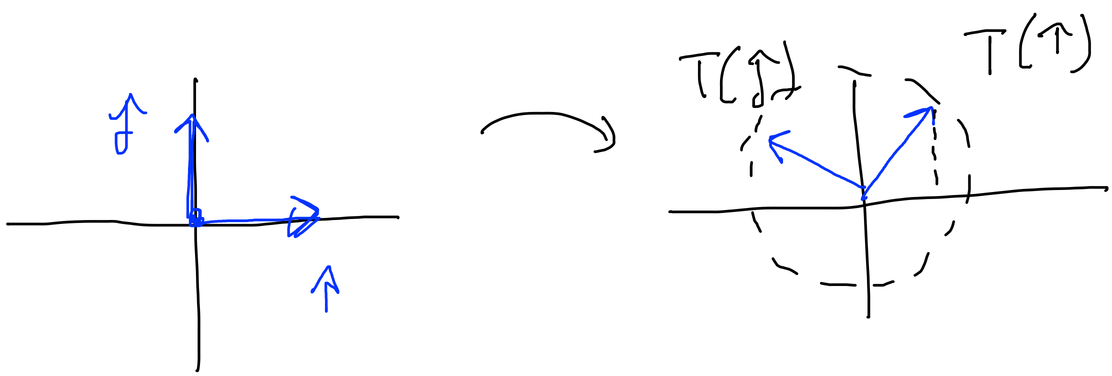
Key find A using $T(\vec{e}_j)$ as columns.

Write $A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}_{m \times n}$

($\Leftarrow$) exercise. (easy). $\square$

Visualizing Linear Transformations

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$



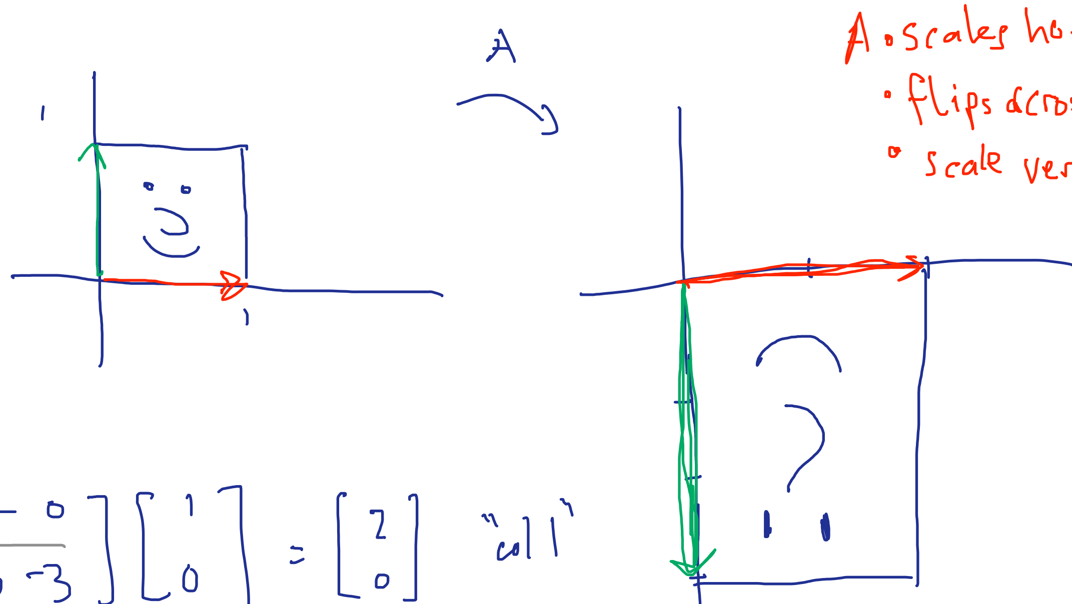
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ? \\ \vdots \\ ? \end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos(\pi/4)\\ \sin(\pi/4)\end{bmatrix} = \begin{bmatrix}\sqrt{2}/2\\ \sqrt{2}/2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\cos(\pi/4)\\ \sin(\pi/4)\end{bmatrix} = \begin{bmatrix}-\sqrt{2}/2\\ \sqrt{2}/2\end{bmatrix}.$$

Ex $A = \begin{bmatrix}2 & 0\\ 0 & -3\end{bmatrix}$ Determine effects of A on $\mathbb{R}^2$ & basis box.

A · scales horizontally by +2
· flips across x-axis
· scale vertically by +3



$$\begin{bmatrix}2 & 0\\ 0 & -3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} \quad \text{"col 1"}$$

$$\begin{bmatrix}2 & 0\\ 0 & -3\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-3\end{bmatrix} \quad \text{"col 2"}$$

<u>Ex</u> $A = \begin{bmatrix} -5/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -5 & 3 \\ 3 & 1 \end{bmatrix}$

$\frac{1}{2}\begin{bmatrix} -5 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -5 \\ 3 \end{bmatrix}$

$\frac{1}{2}\begin{bmatrix} -5 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$