2.1 Mapping Spaces: Introduction to Linear Transformations

function b/w Euclidean spaces! $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

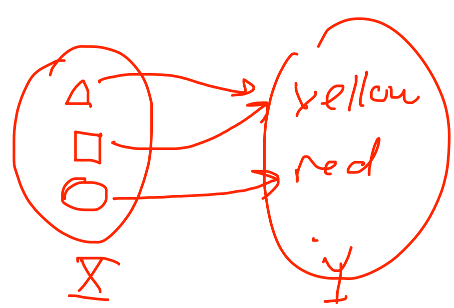
Definition: Let X and Y be any two sets. A ***function*** $f : X \to Y$ is a rule (or a recipe, or a formula) that receives as its input an element x of X, and assigns to x as its output a ***unique*** element y of Y.

We write $y = f(x)$, as usual, and also call y the ***image*** of x under f.

We call X the ***domain*** of f (D(f)), and call Y the ***codomain*** of f. We can also call f a ***map*** (because it tells us where to go), and say that f ***maps*** X ***into*** Y.

$f: X \to Y$ $\quad$ $f: X \xrightarrow{\text{"map"}} Y$

$x \in X \longmapsto f(x) = y \in Y$



$D(f) \subseteq X$, $CoD(f) = \{y \in Y : y = f(x)\}$ "range f"

Linear Transformations

maps of Euclidean spaces

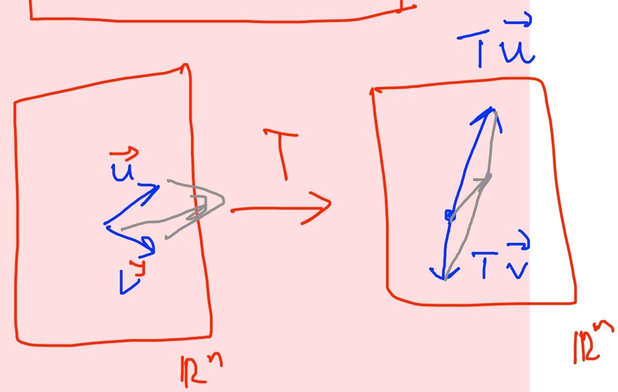
$L : \mathbb{R}^n \to \mathbb{R}^m$

properties

map

Definition: A ***linear*** ***transformation*** $\boxed{T : \mathbb{R}^n \to \mathbb{R}^m}$ is a function that satisfies:

For all $\vec{u}, \vec{v} \in \mathbb{R}^n$, and for all $k \in \mathbb{R}$:



The Additivity Property:

$$\boxed{T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})}$$

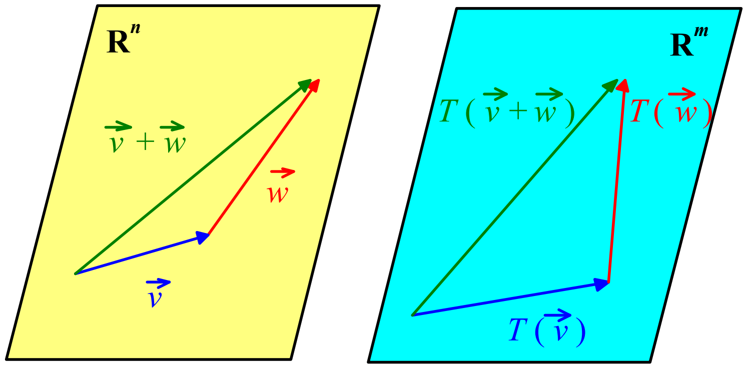
The Homogeneity Property:

$$\boxed{T(k \cdot \vec{u}) = k \cdot T(\vec{u})}$$

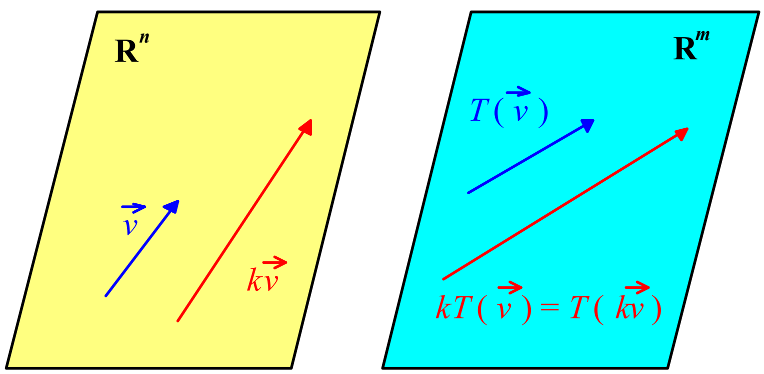
In the special case when $T : \mathbb{R}^n \to \mathbb{R}^n$, that is, the domain is the same space as the codomain, we call T a ***linear operator***.

T = "straight function"

Linear Operator : $T : \mathbb{R}^n \to \mathbb{R}^n$



The Additivity Property



The Homogeneity Property

KEY results in LA.

Theorem: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation ***if and only if*** we can find an $m \times n$ matrix A so that the action of T can be performed by matrix multiplication:

$$\boxed{T(\vec{x}) = A\vec{x}}$$

where we view $\vec{x} \in \mathbb{R}^n$ as an $n \times 1$ matrix. We refer to A as the ***standard matrix*** of T, and we write:

$$[T] = A = \left[T(\vec{e}_1) \mid T(\vec{e}_2) \mid \dots \mid T(\vec{e}_n) \right].$$

In particular, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an ***operator***, $[T]$ is an $n \times n$ or ***square*** matrix.

Some Basic Examples

Example: ***The zero transformation of*** $\mathbb{R}^n$ ***into*** $\mathbb{R}^m$:

$$Z_{n,m} : \mathbb{R}^n \to \mathbb{R}^m, \quad \text{given by}$$

$$\boxed{Z_{n,m}(\vec{x}) = \vec{0}_m} \qquad \text{for all } \vec{x} \in \mathbb{R}^n,$$

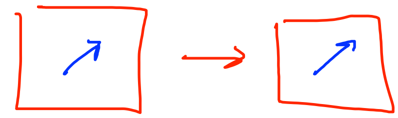
$$[Z_{n,m}] = 0_{m\times n} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For all $\vec{x} \in \mathbb{R}^n$:

$$0_{m\times n}\vec{x} = \vec{0}_m.$$

Example: *The Identity Operator on* $\mathbb{R}^n$:



$$I_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n \quad \text{defined by}$$

$$I_n(\vec{x}) = \boxed{I_{\mathbb{R}^n}(\vec{x}) = \vec{x}} \qquad \text{for all } x \in \mathbb{R}^n$$

The Identity matrix I_n :

$$[I_{\mathbb{R}^n}] = I_n = \left[\vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n\right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$I_3(\vec{x}) = \vec{x}$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

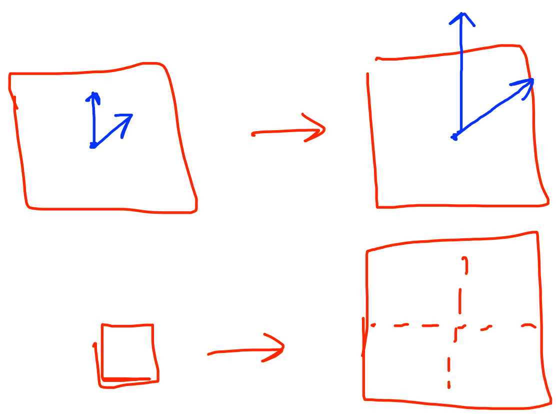
$$I_n \vec{x} = \vec{x}.$$

Example: ***The Scaling Operators:*** For any $k \in \mathbb{R}$:

$$S_k : \mathbb{R}^n \to \mathbb{R}^n,$$

$$\boxed{S_k(\vec{x}) = k\vec{x}} \qquad \text{for all } \vec{x} \in \mathbb{R}^n$$

$k=2$



$[S_k]$ = matrix presentation of S_k.

$[S_k]\vec{x} = k\vec{x}$:

$$\begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} k x_1 \\ k x_2 \\ \vdots \\ k x_n \end{bmatrix}$$

$$[S_k] = k \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = k \cdot I_n$$

Elementary Matrices

Definition: An $\boxed{n \times n}$ matrix E is called an ***elementary matrix*** if it is obtained by performing a $\boxed{\textit{\textbf{single}}}$ elementary row operation on the identity matrix I_n.

EROs

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$ elementary matrix? NO 2 EROs!

Type:	*Notation:*
1. Multiply row i by $c \neq 0$	$R_i \longrightarrow cR_i$
2. Exchange row i and row j	$R_i \longleftrightarrow R_j$
3. Add c times row j to row i	$R_i \longrightarrow R_i + cR_j$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

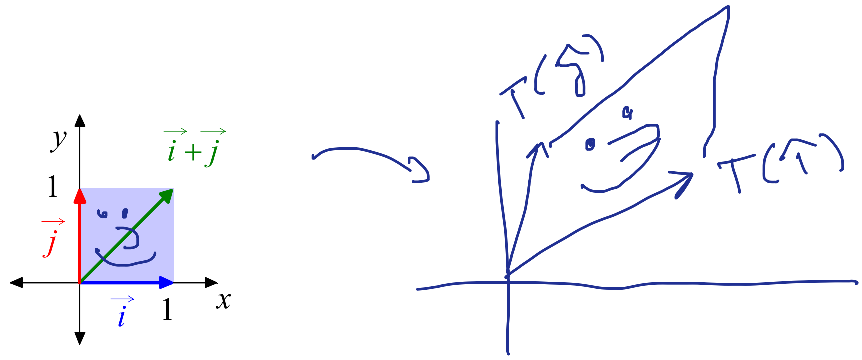
Type I: $3R_2 \to R_2$

Type II: $R_1 \leftrightarrow R_3$

Type III: $R_2 - 3R_1 \to R_2$

The Basic Box

$T : \mathbb{R}^2 \to \mathbb{R}^2$



The Basic Box

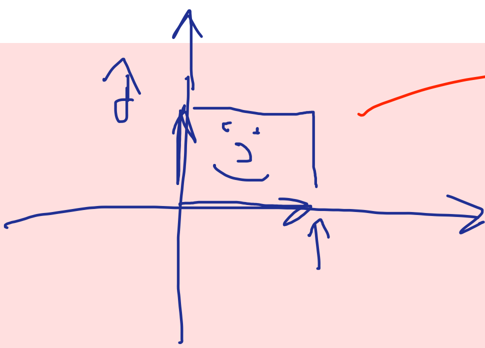
Made of the 3 vectors:

$$\left\{\vec{i},\ \vec{j},\ \vec{i}+\vec{j}\right\}$$

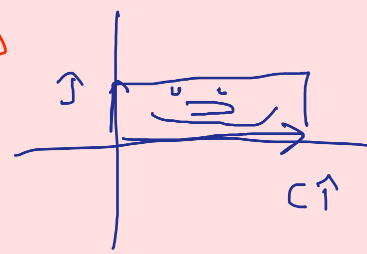
Horizontal and Vertical Dilations and Contractions

A 2×2 Type 1 elementary matrix has the form:

$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

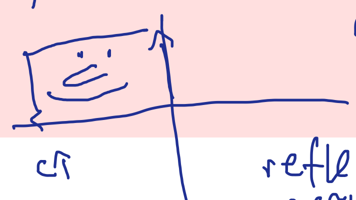


$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$$



$c>0$
scale horiz

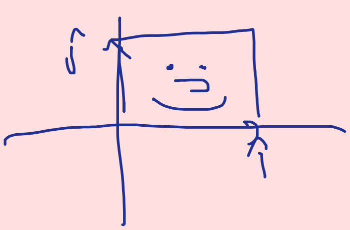
$c<0$



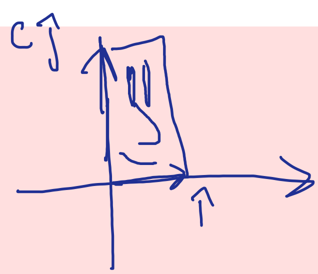
Horizontal Effect

$c>0$

reflection across y-axis & scale horiz.



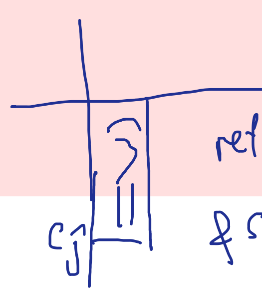
$$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$$



$c>0$
scale vert.

$c<0$

Vertical Effect



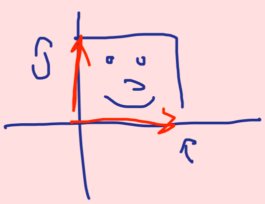
reflect across x-axis & scale vert

$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ c \end{bmatrix}$

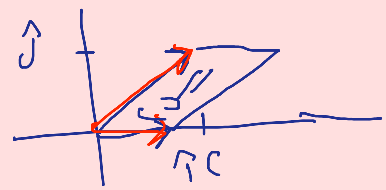
Shear Operators

A 2×2 Type 3 elementary matrix has the form:

$$\begin{bmatrix}1 & c\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \ \& \ \begin{bmatrix}1 & c\\0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}c\\1\end{bmatrix} \qquad \boxed{c>0}$$

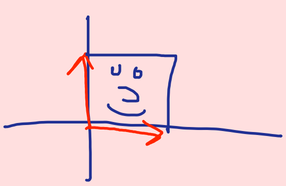


$$\begin{bmatrix}1 & c\\0 & 1\end{bmatrix}$$

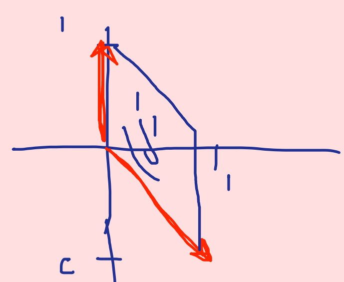


A Horizontal Shearing Operator

$$\begin{bmatrix}1 & 0\\c & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\c\end{bmatrix} \ \& \ \begin{bmatrix}1 & 0\\c & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \qquad \boxed{c<0}$$



$$\begin{bmatrix}1 & 0\\c & 1\end{bmatrix}$$



A Vertical Shearing Operator

* no flips.