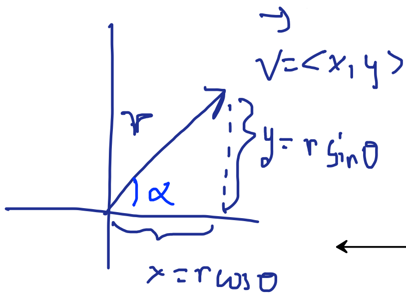
2.2 Rotations, Projections and Reflections

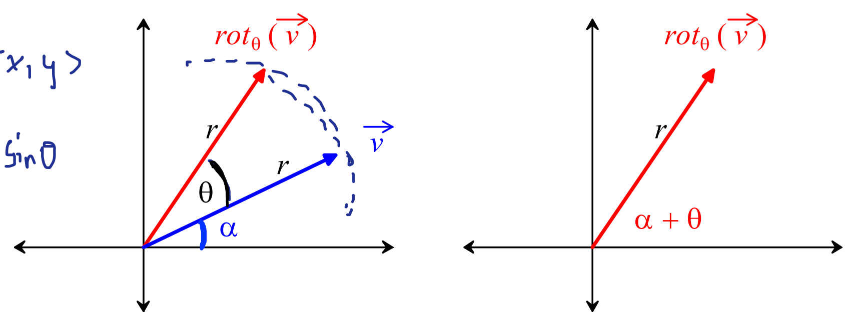
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\overset{\vec{v}}{\begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}} = \begin{bmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix}$$

Rotations in $\mathbb{R}^2$

trig ID

$$= \begin{bmatrix} r(\cos\alpha\cos\theta - \sin\alpha\sin\theta) \\ r(\sin\alpha\cos\theta + \cos\alpha\sin\theta) \end{bmatrix}_{2\times 1}$$





A Vector $\vec{v}$ and $\boxed{rot_\theta(\vec{v})}$,

its Counterclockwise Rotation by θ

$$= \begin{bmatrix} (\cos\theta)(r\cos\alpha) + (-\sin\theta)(r\sin\alpha) \\ (\cos\theta)(r\sin\alpha) + (\sin\theta)(r\cos\alpha) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\overset{\vec{v}}{\begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}}$$

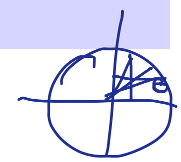
$$= \boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$Rot_\theta(\vec{v}) = \langle x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta \rangle$$

Theorem: The function $rot_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ that takes a vector $\vec{v}$ and rotates $\vec{v}$ counterclockwise by an angle of θ about the origin is a ***linear transformation***, with:

$$[rot_\theta] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Note
θ is fixed!
so LT!

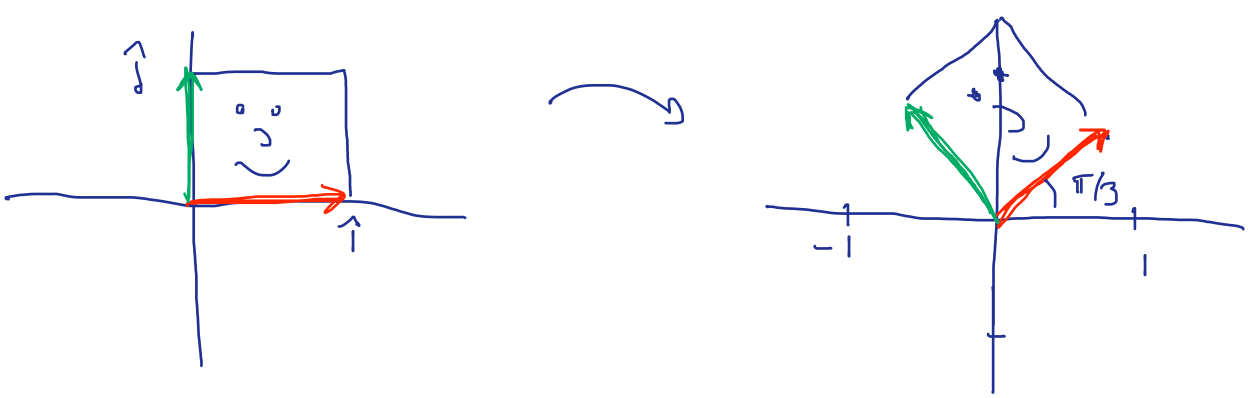


$\theta = \pi/6 = 30^\circ$

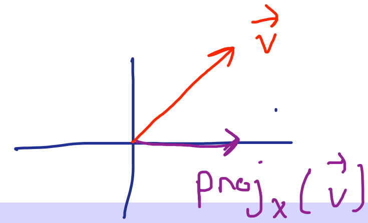
$$[rot_{\pi/6}] = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} \quad \& \quad \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

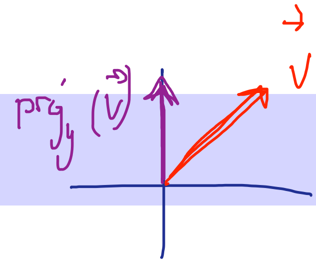
$\sqrt{3}/2 \approx 0.866$ $\quad \approx \begin{bmatrix} 0.9 \\ 0.5 \end{bmatrix}$ $\quad \approx \begin{bmatrix} -0.5 \\ 0.9 \end{bmatrix}$



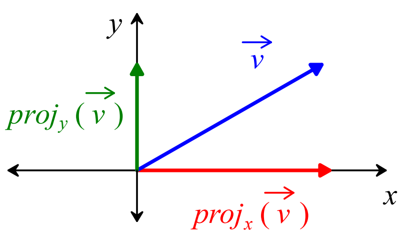
Basic Projections in $\mathbb{R}^2$

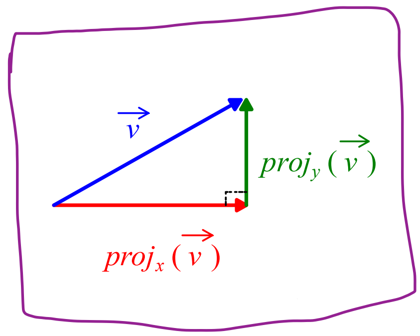


$$\boxed{proj_x(\langle x,y\rangle) = \langle x,0\rangle.}$$



$$\boxed{proj_y(\langle x,y\rangle) = \langle 0,y\rangle.}$$





Key Relationship:

$$\boxed{\begin{aligned}\vec{v} &= \langle x,y\rangle \\ &= \langle x,0\rangle + \langle 0,y\rangle \\ &= proj_x(\vec{v}) + proj_y(\vec{v})\end{aligned}}$$

$$[Proj_x]\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\0\end{bmatrix}$$

$$\|$$

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\0\end{bmatrix}$$

$$[Proj_y]\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\y\end{bmatrix}$$

$$\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\y\end{bmatrix}$$

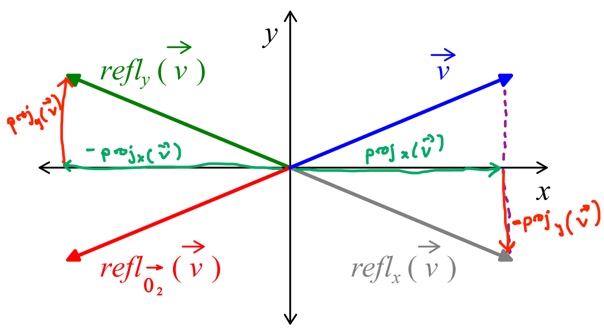
$$proj_x(\vec{v}) = \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ and}$$

$$proj_y(\vec{v}) = \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[proj_x] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[proj_y] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Basic Reflections in $\mathbb{R}^2$



A Vector $\vec{v}$ and its Three Basic Reflections in $\mathbb{R}^2$.

across x-axis
across y-axis

- Reflection Across x-axis

$$\boxed{Refl_x(\vec{v}) = proj_x(\vec{v}) - proj_y(\vec{v})}$$

$$\boxed{[Refl_x] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

Check $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} x \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} x-0 \\ 0-y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

- Reflection Across y-axis

$$Refl_y(\vec{v}) = \begin{bmatrix} -x \\ y \end{bmatrix} = -proj_x(\vec{v}) + proj_y(\vec{v})$$

$$\boxed{[Refl_y] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$-\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

$$refl_x\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$$

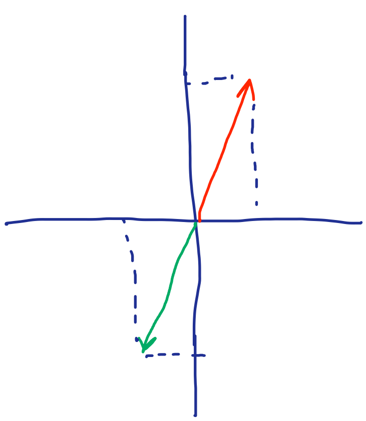
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

$$refl_y\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}$$

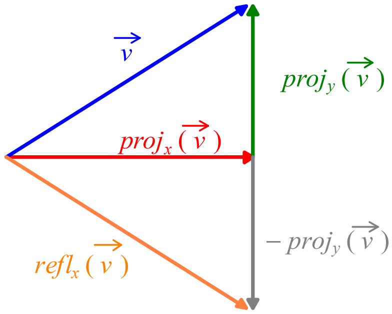
$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and}$$

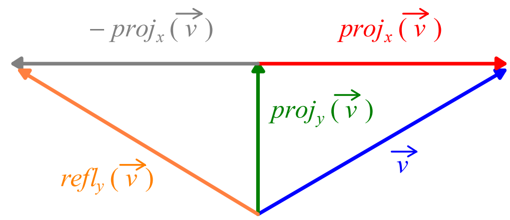
Across the origin

$$refl_{\vec{0}_2}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \end{bmatrix}$$



$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$





The Geometric Relationships Among $\vec{v}$, $proj_x(\vec{v})$, $proj_y(\vec{v})$, $refl_x(\vec{v})$ and $refl_y(\vec{v})$

$$\boxed{refl_x(\vec{v}) = proj_x(\vec{v}) - proj_y(\vec{v}),} \text{ and}$$

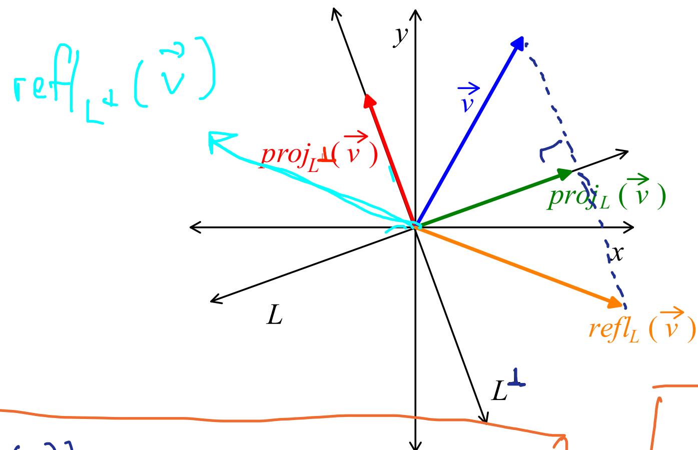
$$\boxed{refl_y(\vec{v}) = proj_y(\vec{v}) - proj_x(\vec{v}).}$$

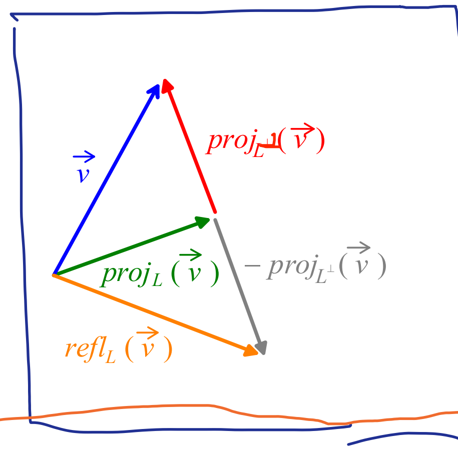
General Projections, and Reflections in $\mathbb{R}^2$

L: line through origin

$L^{\perp}$: line through origin perpendicular to L

$$\boxed{\vec{v} = \text{proj}_L(\vec{v}) + \text{proj}_{L^{\perp}}(\vec{v})}$$



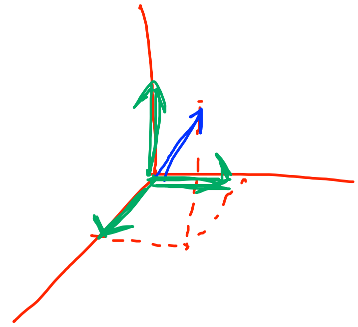


$$\boxed{\text{Refl}_{L^{\perp}}(\vec{v}) = -\text{proj}_L(\vec{v}) + \text{proj}_{L^{\perp}}(\vec{v})}$$

$$\boxed{\text{Refl}_L(\vec{v}) = \text{proj}_L(\vec{v}) - \text{proj}_{L^{\perp}}(\vec{v})}$$

The Projections of $\vec{v}$ Onto a Line L and its Orthogonal Complement $L^{\perp}$, and the Reflection of $\vec{v}$ Across L.

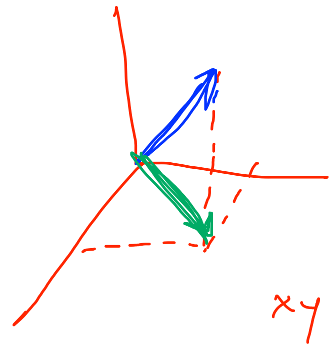
Basic Projections in $\mathbb{R}^3$



$$\boxed{proj_x(\langle x,y,z\rangle) = \langle x,0,0\rangle}$$

$$proj_y(\langle x,y,z\rangle) = \langle 0,y,0\rangle$$

$$proj_z(\langle x,y,z\rangle) = \langle 0,0,z\rangle$$



$$\boxed{proj_{xy}(\langle x,y,z\rangle) = \langle x,y,0\rangle}$$

$$proj_{xz}(\langle x,y,z\rangle) = \langle x,0,z\rangle$$

$$proj_{yz}(\langle x,y,z\rangle) = \langle 0,y,z\rangle$$

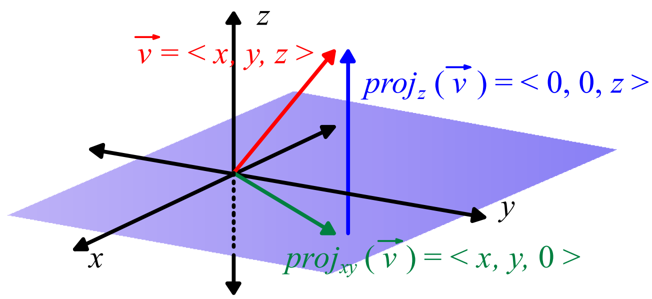
$$[proj_x]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$$

$$[proj_{xy}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
\vec{v} &= \langle x, y, z \rangle \\
&= \langle x, 0, 0 \rangle + \langle 0, y, z \rangle \\
&= proj_x(\langle x, y, z \rangle) + proj_{yz}(\langle x, y, z \rangle),
\end{aligned}$$

$$\begin{aligned}
\vec{v} &= \langle x, y, z \rangle \\
&= \langle 0, y, 0 \rangle + \langle x, 0, z \rangle \\
&= proj_y(\langle x, y, z \rangle) + proj_{xz}(\langle x, y, z \rangle), \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\vec{v} &= \langle x, y, z \rangle \\
&= \langle 0, 0, z \rangle + \langle x, y, 0 \rangle \\
&= proj_z(\langle x, y, z \rangle) + proj_{xy}(\langle x, y, z \rangle)
\end{aligned}$$



The Relationships Among $\vec{v}$, $proj_{xy}(\vec{v})$ and $proj_z(\vec{v})$.

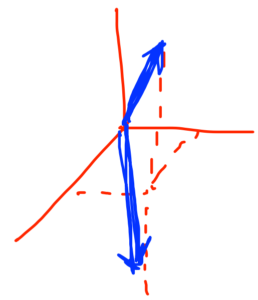
$$\begin{aligned} refl_{xy}(\langle x,y,z\rangle) &= \langle x,y,-z\rangle \\ &= \langle x,y,0\rangle - \langle 0,0,z\rangle \\ &= proj_{xy}(\langle x,y,z\rangle) - proj_z(\langle x,y,z\rangle). \\ refl_z(\langle x,y,z\rangle) &= proj_z(\langle x,y,z\rangle) - proj_{xy}(\langle x,y,z\rangle) \\ &= \langle 0,0,z\rangle - \langle x,y,0\rangle \\ &= \langle -x,-y,z\rangle. \end{aligned}$$

The Basic *Reflection Operators*:

$$refl_x(\langle x,y,z\rangle) = \langle x,-y,-z\rangle$$

$$refl_y(\langle x,y,z\rangle) = \langle -x,y,-z\rangle$$

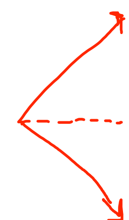
$$refl_z(\langle x,y,z\rangle) = \langle -x,-y,z\rangle$$



$$\boxed{refl_{xy}(\langle x,y,z\rangle) = \langle x,y,-z\rangle}$$

$$refl_{xz}(\langle x,y,z\rangle) = \langle x,-y,z\rangle$$

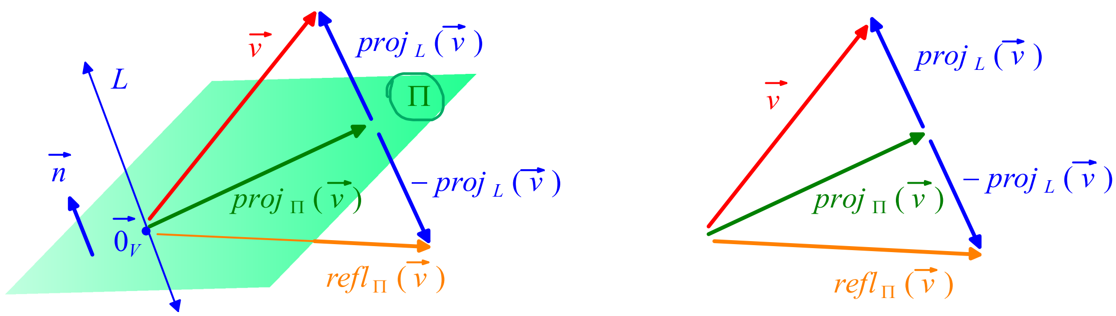
$$refl_{yz}(\langle x,y,z\rangle) = \langle -x,y,z\rangle$$



$$refl_{\vec{0}_3}(\langle x,y,z\rangle) = \langle -x,-y,-z\rangle$$

$$[refl_{xy}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

General Projections and Reflections in $\mathbb{R}^3$



We need the decomposition:

key relationship.

$$\boxed{\vec{v} = proj_\Pi(\vec{v}) + proj_L(\vec{v}),}$$

where $proj_\Pi(\vec{v}) \in \Pi$ and $proj_L(\vec{v}) \in L$.

Here $L = \Pi^\perp$

General Principle:

$$\text{If } \vec{v} = proj_W(\vec{v}) + proj_{W^\perp}(\vec{v}), \text{ then:}$$

$$refl_W(\vec{v}) = proj_W(\vec{v}) - proj_{W^\perp}(\vec{v})$$

For planes Π and lines L through the origin:

$$refl_\Pi(\vec{v}) = proj_\Pi(\vec{v}) - proj_L(\vec{v}), \text{ and}$$

$$refl_L(\vec{v}) = proj_L(\vec{v}) - proj_\Pi(\vec{v}).$$